Signal Processing Structures For Optimization

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Signal processing structures

$$f^{(0)}[n] = r_{ss}[n]$$
$$g^{(0)}[n] = r_{ss}[n-1]$$

for $\quad i = 1, 2, \ldots, p$

$$k_i = \frac{f^{(i-1)}[i]}{g^{(i-1)}[i]}$$
$$f^{(i)}[n] = f^{(i-1)}[n] - k_i g^{(i-1)}[n]$$
$$g^{(i)}[n] = g^{(i-1)}[n-1] - k_i f^{(i-1)}[n-1]$$

end

Given $k_1, k_2, \ldots, k_p$

for $i = 1, 2, \ldots, p$

$$\alpha_i^{(i)} = k_i$$

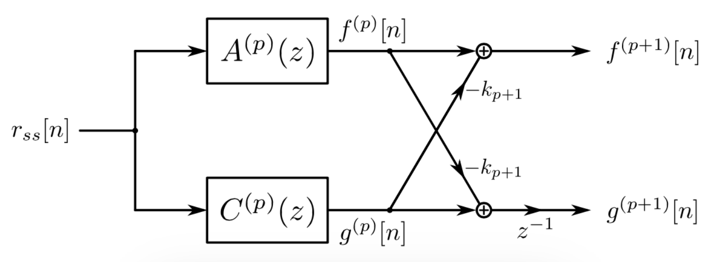
if $i > 1$ then for $j = 1, 2, \ldots, i-1$

$$\alpha_j^{(j)} = \alpha_j^{(i-1)} - k_i \alpha_{i-j}^{(i-1)}$$

end

end

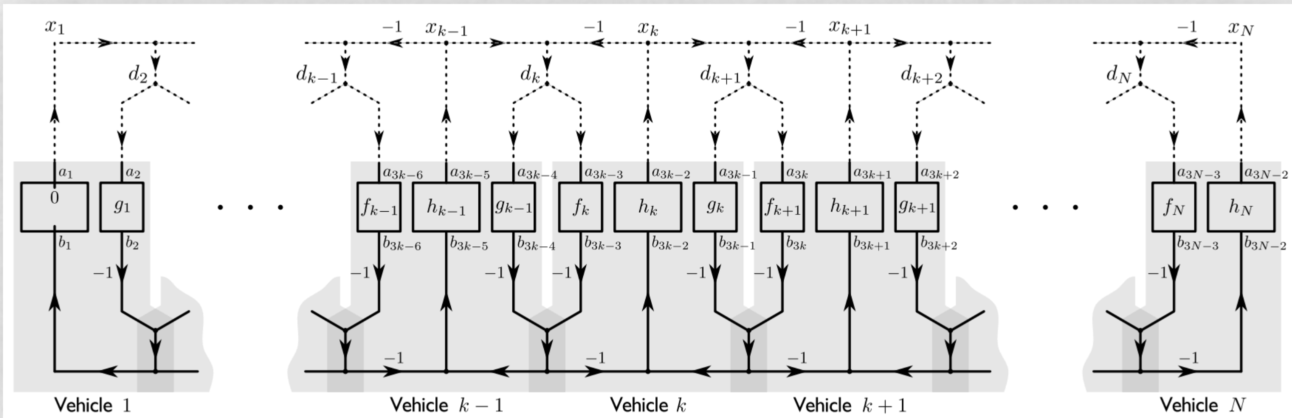
vs.

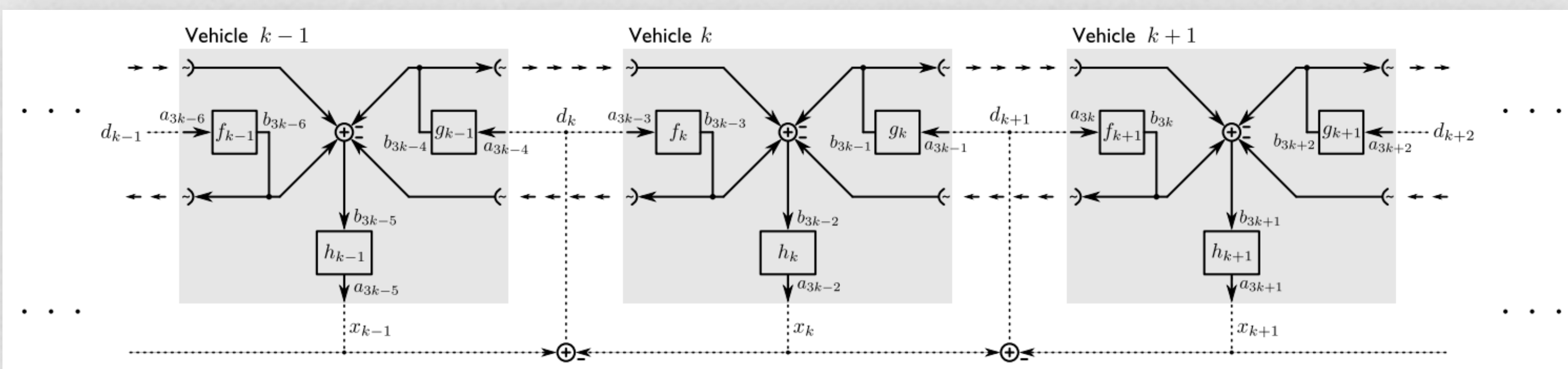


$$k_{p+1} = \frac{f^{(p)}[p+1]}{g^{(p)}[p+1]}.$$

From: T. A. Baran and A. V. Oppenheim, "A Derivation of the Recursive Solution to the Autocorrelation Normal Equations [Lecture Notes]," *IEEE Signal Processing Magazine*, January 2013.

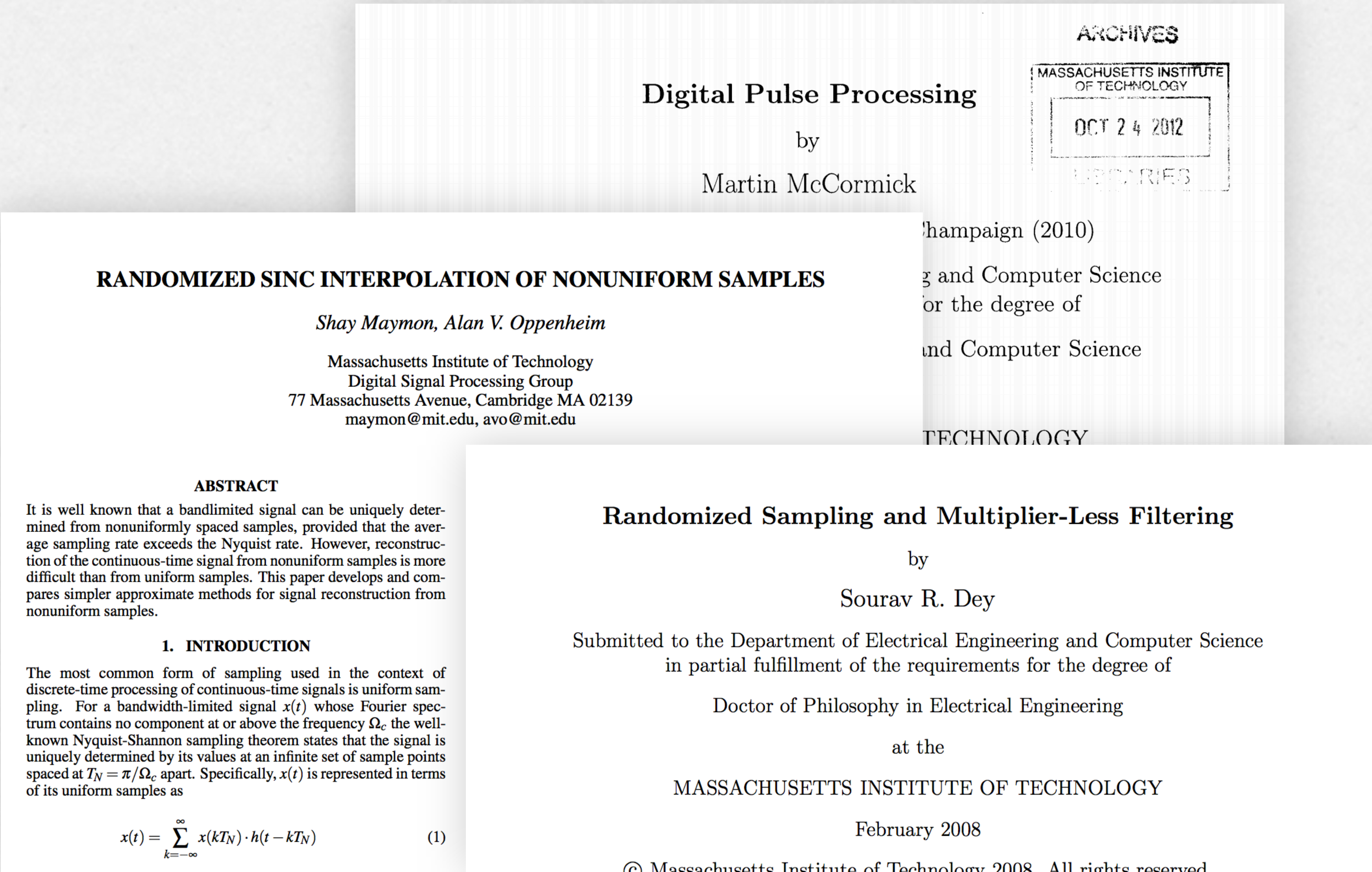
Signal processing structures





From: T. A. Baran and B. K. P. Horn, "A Robust Signal-Flow Architecture for Cooperative Vehicle Density Control," in Proceedings of the IEEE ICASSP (Vancouver, British Columbia, Canada), May 26 - May 31, 2013.

Signal processing structures



Digital Pulse Processing

by

Martin McCormick

...hampaign (2010)

...g and Computer Science
...or the degree of

...and Computer Science

...TECHNOLOGY

RANDOMIZED SINC INTERPOLATION OF NONUNIFORM SAMPLES

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ABSTRACT

It is well known that a bandlimited signal can be uniquely determined from nonuniformly spaced samples, provided that the average sampling rate exceeds the Nyquist rate. However, reconstruction of the continuous-time signal from nonuniform samples is more difficult than from uniform samples. This paper develops and compares simpler approximate methods for signal reconstruction from nonuniform samples.

1. INTRODUCTION

The most common form of sampling used in the context of discrete-time processing of continuous-time signals is uniform sampling. For a bandwidth-limited signal $x(t)$ whose Fourier spectrum contains no component at or above the frequency Ω_c the well-known Nyquist-Shannon sampling theorem states that the signal is uniquely determined by its values at an infinite set of sample points spaced at $T_N = \pi/\Omega_c$ apart. Specifically, $x(t)$ is represented in terms of its uniform samples as

$$x(t) = \sum_{k=-\infty}^{\infty} x(kT_N) \cdot h(t - kT_N) \quad (1)$$

Randomized Sampling and Multiplier-Less Filtering

by

Sourav R. Dey

Submitted to the Department of Electrical Engineering and Computer Science
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Electrical Engineering

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

February 2008

Opening demo

$$\min_{\underline{\mathrm{x}},\underline{\mathrm{e}},\underline{\mathrm{b}}} \; ||\underline{\mathrm{x}}||_1 + \frac{\rho}{2}||\underline{\mathrm{e}}||_2^2$$

$$\text{s.t.} \quad \underline{\mathrm{e}} = A\underline{\mathrm{x}} - \underline{\mathrm{b}}$$

$$\underline{\mathrm{b}} = \hat{\underline{\mathrm{b}}}$$

Opening demo

$$\min_{\underline{x},\underline{e},\underline{b}} \; ||\underline{x}||_1 + \frac{\rho}{2}||\underline{e}||_2^2$$

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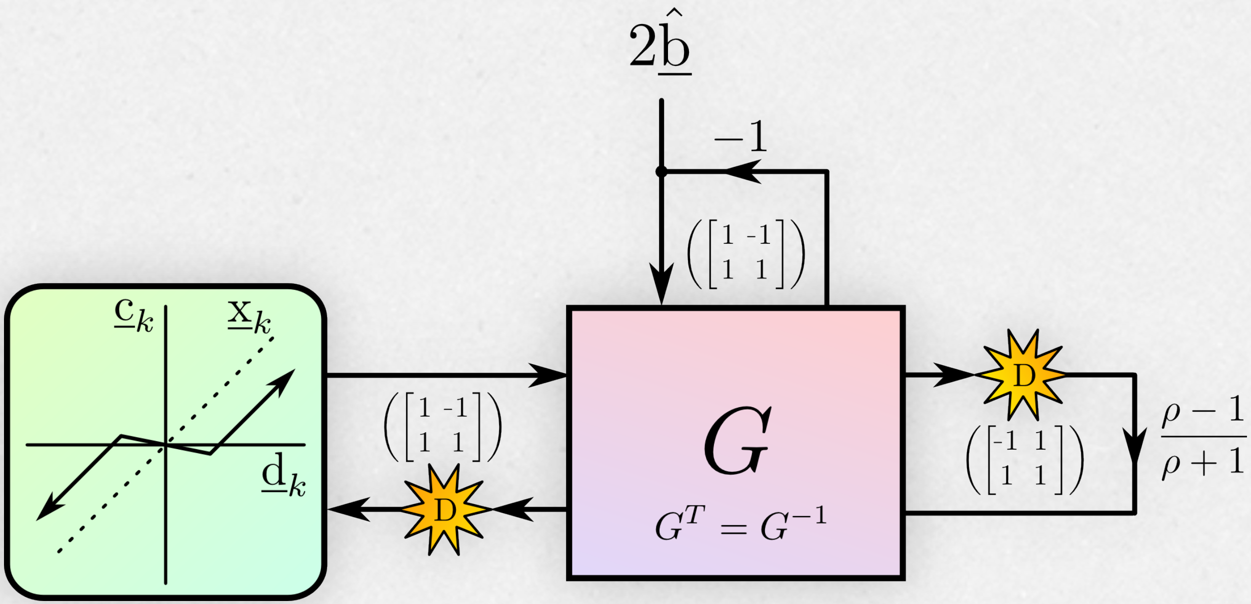

Opening demo

$$\min_{\underline{x},\underline{e},\underline{b}} \; ||\underline{x}||_1 + \frac{\rho}{2}||\underline{e}||_2^2$$

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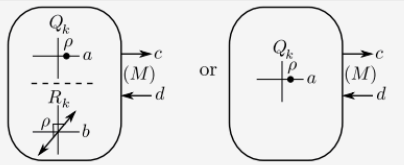
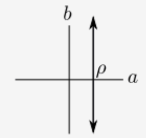
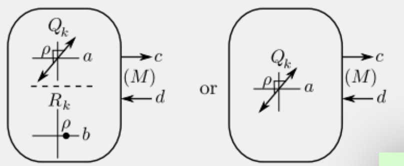
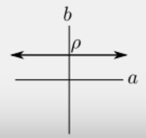
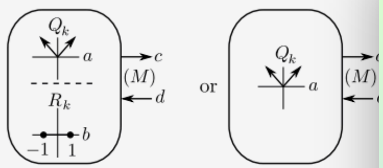
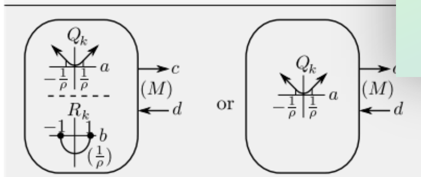
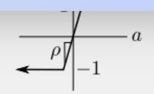
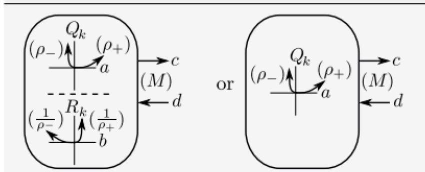
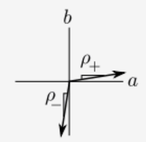
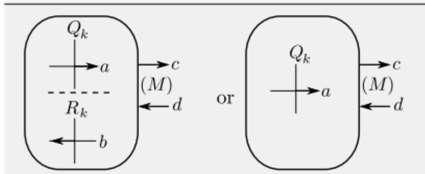
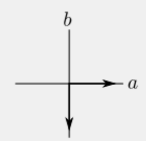
$$\min_{\underline{x},\underline{e},\underline{b}} \; ||\underline{x}||_1 + \frac{\rho}{2}||\underline{e}||_2^2$$

$$\text{s.t.} \quad \underline{e} = A\underline{x} - \underline{b}$$

$$\underline{b} = \underline{\hat{b}}$$

Overview

Symbol	Reduced-form primal components	Reduced-form dual components	Behavior (canonical coordinates)	Transformation matrix	Realization as a map
	$\widehat{Q}_k(a) = 0$ $a = \rho$	$\widehat{R}_k(b) = \rho b$ $b \in \mathbb{R}$		$M_k^{(CR)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $M_k^{(CR)} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	$c = -d + 2\rho$ $c = d - 2\rho$
	$\widehat{Q}_k(a) = \rho a$ $a \in \mathbb{R}$	$\widehat{R}_k(b) = 0$ $b = \rho$		$M_k^{(CR)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $M_k^{(CR)} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	$c = d - 2\rho$ $c = -d + 2\rho$
	[illegible]	[illegible]	[illegible]	$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	$c = \begin{cases} d+2, & d < -1 \\ -d, & \lvert d\rvert \le 1 \\ d-2, & d > 1 \end{cases}$ $c = \begin{cases} -d-2, & d < -1 \\ d, & \lvert d\rvert \le 1 \\ -d+2, & d > 1 \end{cases}$
	[illegible] $\frac{1}{2}\rho a^2 + \frac{1}{2\rho} \quad \lvert a\rvert < \frac{1}{\rho}$ $a \in \mathbb{R}$	[illegible] $-\frac{1}{2\rho}$ $-1 \le b \le 1$		$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $M_k^{(CR)} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	$c = \begin{cases} \frac{1-\rho}{1+\rho}d, & \lvert d\rvert \le \frac{1}{\rho}+1 \\ d-2, & d > \frac{1}{\rho}+1 \\ d+2, & d < -\frac{1}{\rho}-1 \end{cases}$ $c = \begin{cases} \frac{\rho-1}{\rho+1}d, & \lvert d\rvert \le \frac{1}{\rho}+1 \\ -d+2, & d > \frac{1}{\rho}+1 \\ -d-2, & d < -\frac{1}{\rho}-1 \end{cases}$
	$\widehat{Q}_k(a) = \begin{cases} \frac{1}{2}\rho_+ a^2 & a \ge 0 \\ \frac{1}{2}\rho_- a^2 & a < 0 \end{cases}$ $a \in \mathbb{R}$	$\widehat{R}_k(b) = \begin{cases} \frac{1}{2}\frac{1}{\rho_+}b^2 & b \ge 0 \\ \frac{1}{2}\frac{1}{\rho_-}b^2 & b < 0 \end{cases}$ $b \in \mathbb{R}$		$M_k^{(CR)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $M_k^{(CR)} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	$c = \begin{cases} \frac{1-\rho_+}{1+\rho_+}d, & d \ge 0 \\ \frac{1-\rho_-}{1+\rho_-}d, & d < 0 \end{cases}$ $c = \begin{cases} \frac{\rho_+-1}{\rho_++1}d, & d \ge 0 \\ \frac{\rho_--1}{\rho_-+1}d, & d < 0 \end{cases}$
	$\widehat{Q}_k(a) = 0$ $a \ge 0$	$\widehat{R}_k(b) = 0$ $b \le 0$		$M_k^{(CR)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $M_k^{(CR)} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	$c = \lvert d\rvert$ $c = -\lvert d\rvert$

Where we're going to end up…

Overview

Signal Processing Structures

Stationarity Conditions

Conservation

Overview

1.

Signal Processing Structures

2.

Stationarity Conditions

3.

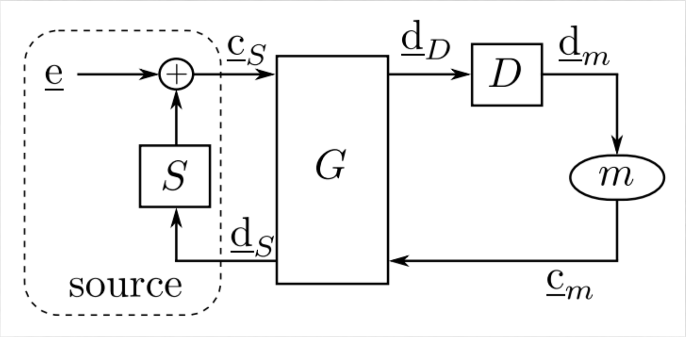
Conservation

4.

Examples

A class of signal processing structures

A class of signal processing structures

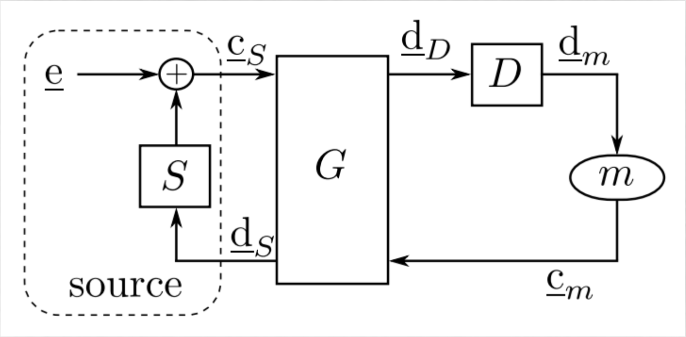


$S^T = S^{-1}$, $G^T = G^{-1}$

m: a memoryless nonlinearity

A class of signal processing structures

A solution $\underline{c}^*$, $\underline{d}^*$ is known to exist when $\underline{d}_D = \underline{d}_m$



$S^T = S^{-1}$, $G^T = G^{-1}$

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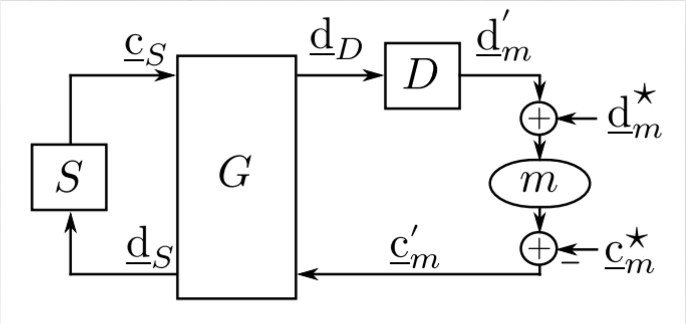
A class of signal processing structures

A solution $\underline{c}^*$, $\underline{d}^*$ is known to exist when $\underline{d}_D = \underline{d}_m$

$$\underline{d}_m = \underline{d}'_m + \underline{d}^*_m$$

$$\underline{c}_m = \underline{c}'_m + \underline{c}^*_m$$



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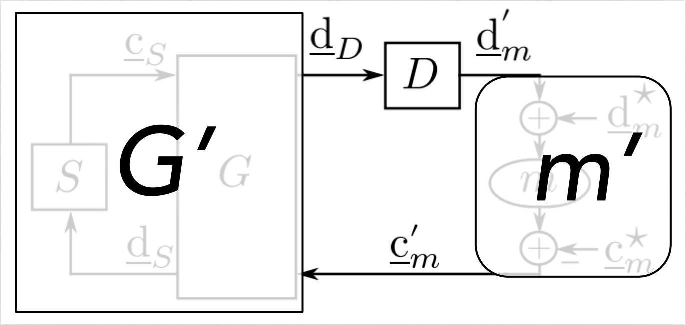


$G'^T = G'^{-1}$

m': a memoryless nonlinearity

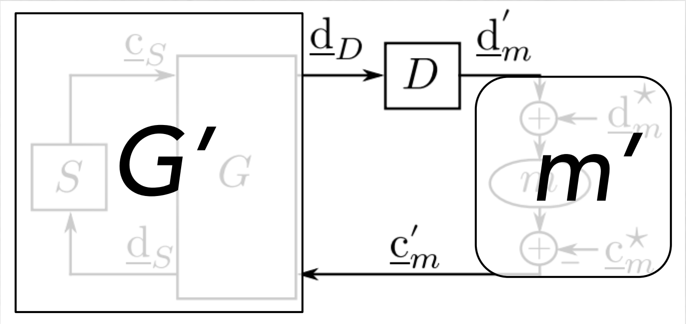
A class of signal processing structures

A solution $\underline{c}^*$, $\underline{d}^*$ is known to exist when $\underline{d}_D = \underline{d}_m$

If $G'm'(\cdot)$ is contractive ($\|\underline{d}_D\|_2 \leq \alpha \|\underline{d}'_m\|_2$, $\alpha < 1$)…

$$\underline{d}_m = \underline{d}'_m + \underline{d}^*_m$$

$$\underline{c}_m = \underline{c}'_m + \underline{c}^*_m$$



Linear (decaying exponential) convergence.

$$\|\underline{d}'_m[n]\|_2 \leq k\alpha^n$$

A class of signal processing structures

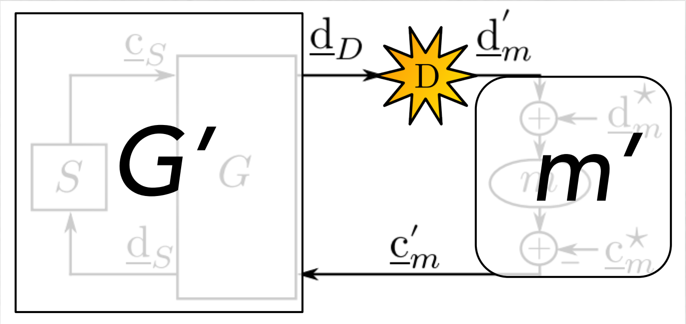
A solution $\underline{c}^*$, $\underline{d}^*$ is known to exist when $\underline{d}_D = \underline{d}_m$

If $G'm'(\cdot)$ is contractive for arbitrary subvector updates: $\underline{d}_D \rightarrow \underline{d}'_m$

One per step

$$\underline{d}_m = \underline{d}'_m + \underline{d}^*_m$$

$$\underline{c}_m = \underline{c}'_m + \underline{c}^*_m$$



Linear (decaying exponential) convergence.

$$\left\| \underline{d}'_m[n] \right\|_2 \leq k\alpha^n$$

A class of signal processing structures

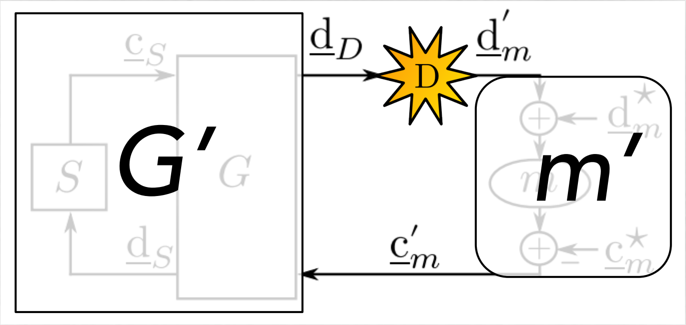
A solution $\underline{c}^*$, $\underline{d}^*$ is known to exist when $\underline{d}_D = \underline{d}_m$

If $G'm'(\cdot)$ is contractive for arbitrary subvector updates: $\underline{d}_D \rightarrow \underline{d}'_m$

Stochastic updates

$$\underline{d}_m = \underline{d}'_m + \underline{d}^*_m$$

$$\underline{c}_m = \underline{c}'_m + \underline{c}^*_m$$



$$E\left[\left\|\underline{d}'_m[n]\right\|_2^2\right] \le x[n]$$

$$x[n] = \alpha^2 \sum_{m=1}^{\infty} x[n-m]\, p(\text{previous firing } m \text{ steps ago})$$

A class of signal processing structures

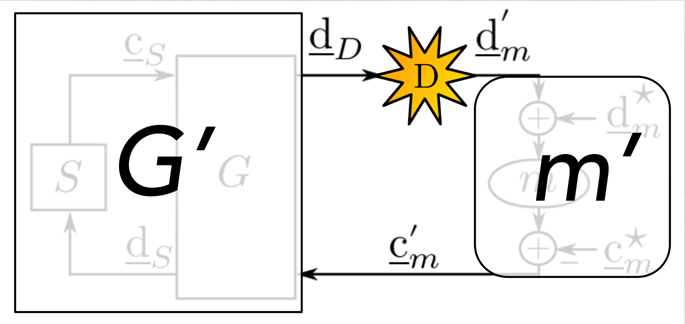
A solution $\underline{c}^*$, $\underline{d}^*$ is known to exist when $\underline{d}_D = \underline{d}_m$

If $G'm'(\cdot)$ is contractive for arbitrary subvector updates: $\underline{d}_D \rightarrow \underline{d}'_m$

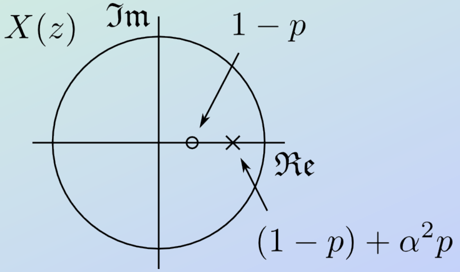
Stochastic updates – **Bernoulli** with probability p

$$\underline{d}_m = \underline{d}'_m + \underline{d}^*_m$$

$$\underline{c}_m = \underline{c}'_m + \underline{c}^*_m$$



$$E\left[\left\|\underline{d}'_m[n]\right\|_2^2\right] \le x[n]$$



Stationarity conditions

Stationarity conditions

Conservation in Signal Processing Systems

by

Thomas A. Baran

B.S. Electrical
B.S. Biomedica
S.M. EECS, Mass

Submitted to the Departme
in partial fulfillme

Doctor of Philosophy in

*Stationary Principles and Potential Functions for Nonlinear Networks**

by L. O. CHUA

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The concepts of total
otal parametric content
The results by Brayton
alized to non-complete
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or which each of these
ion is shown to be the
in terms of standard

CXVI. *Some General Theorems for Non-Linear Systems Possessing Resistance.*

By William Millar,
Atomic Energy Research Establishment, Harwell*.

[Revised MS. received June 8, 1951.]

Stationarity conditions

Primal canonical form:

$$\begin{aligned}\min_{\substack{\{y_1,\ldots,y_N\}\\ \{a_1,\ldots,a_N\}}} \quad & \sum_{k=1}^{K} Q_k(\underline{\mathrm{y}}_k^{(CR)})\\ \text{s.t.} \quad & \underline{\mathrm{a}}_k^{(CR)} = f_k(\underline{\mathrm{y}}_k^{(CR)}), \quad k = 1,\ldots,K\\ & A_\ell \underline{\mathrm{a}}_\ell^{(i)} = \underline{\mathrm{a}}_\ell^{(o)}, \quad \ell = 1,\ldots,L.\end{aligned}$$

$$\nabla Q_k(\underline{\mathrm{y}}_k^{(CR)}) = J_{f_k}^T(\underline{\mathrm{y}}_k^{(CR)}) g_k(\underline{\mathrm{y}}_k^{(CR)}),$$

Stationarity conditions

Primal canonical form:

$$\begin{aligned}
\min_{\substack{\{y_1,\ldots,y_N\}\\ \{a_1,\ldots,a_N\}}} \quad & \sum_{k=1}^{K} Q_k(\underline{\mathrm{y}}_k^{(CR)}) \\
\text{s.t.} \quad & \underline{\mathrm{a}}_k^{(CR)} = \boxed{f_k(\underline{\mathrm{y}}_k^{(CR)}),} \quad k = 1, \ldots, K \\
& \boxed{A_\ell} \underline{\mathrm{a}}_\ell^{(i)} = \underline{\mathrm{a}}_\ell^{(o)}, \quad \ell = 1, \ldots, L.
\end{aligned}$$

$$\nabla Q_k(\underline{\mathrm{y}}_k^{(CR)}) = J_{f_k}^T(\underline{\mathrm{y}}_k^{(CR)}) \boxed{g_k(\underline{\mathrm{y}}_k^{(CR)}),}$$

Stationarity conditions

Primal canonical form:

$$\begin{aligned}\min_{\substack{\{y_1,\ldots,y_N\}\\\{a_1,\ldots,a_N\}}} \quad & \sum_{k=1}^{K} Q_k(\underline{\mathrm{y}}_k^{(CR)}) \\ \text{s.t.} \quad & \underline{\mathrm{a}}_k^{(CR)} = f_k(\underline{\mathrm{y}}_k^{(CR)}), \quad k = 1, \ldots, K \\ & A_\ell \underline{\mathrm{a}}_\ell^{(i)} = \underline{\mathrm{a}}_\ell^{(o)}, \quad \ell = 1, \ldots, L.\end{aligned}$$

$$\nabla Q_k(\underline{\mathrm{y}}_k^{(CR)}) = J_{f_k}^T(\underline{\mathrm{y}}_k^{(CR)}) g_k(\underline{\mathrm{y}}_k^{(CR)}),$$

For example:

$$f_k\left(\underline{y}_k^{(CR)}\right) = \underline{y}_k^{(CR)} \quad g_k\left(\underline{y}_k^{(CR)}\right) \quad => \quad \nabla Q_k\left(\underline{y}_k^{(CR)}\right) = g_k\left(\underline{y}_k^{(CR)}\right)$$

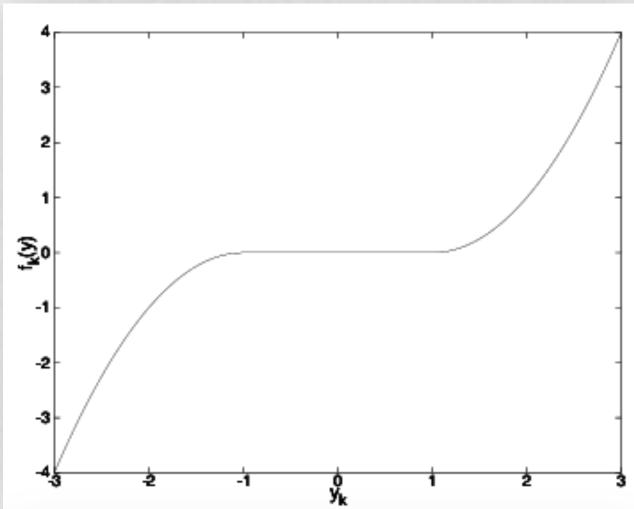
Stationarity conditions

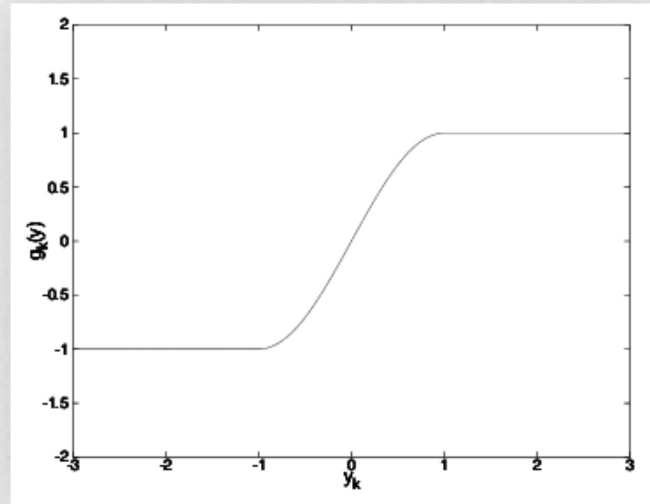
Dual canonical form:

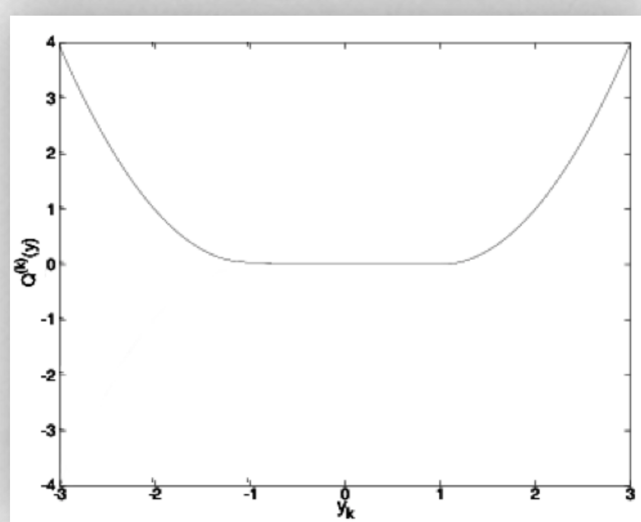
$$\begin{aligned} \max_{\substack{\{\underline{y}_1,\dots,\underline{y}_N\} \\ \{\underline{b}_1,\dots,\underline{b}_N\}}} \quad & -\sum_{k=1}^{K} R_k(\underline{y}_k^{(CR)}) \\ \text{s.t.} \quad & \underline{b}_k = g_k(\underline{y}_k^{(CR)}), \quad k = 1,\dots,K \\ & \underline{b}_\ell^{(i)} = -A_\ell^T \underline{b}_\ell^{(o)}, \quad \ell = 1,\dots,L, \end{aligned}$$

$$R_k(\underline{y}_k^{(CR)}) = \left\langle f_k(\underline{y}_k^{(CR)}), g_k(\underline{y}_k^{(CR)}) \right\rangle - Q_k(\underline{y}_k^{(CR)}), \quad k = 1,\dots,K,$$

Stationarity conditions

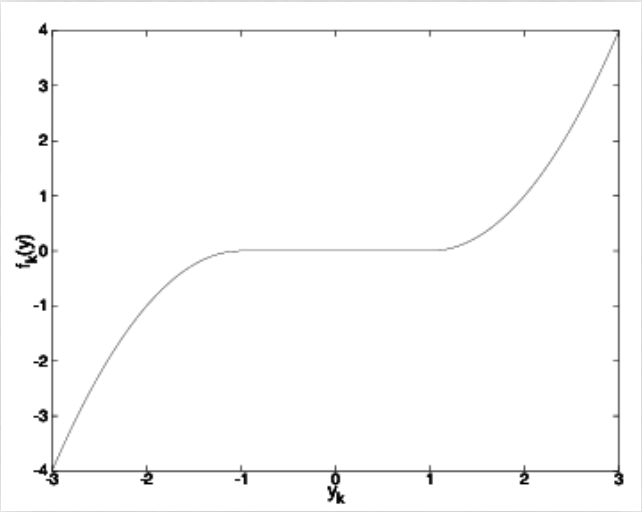


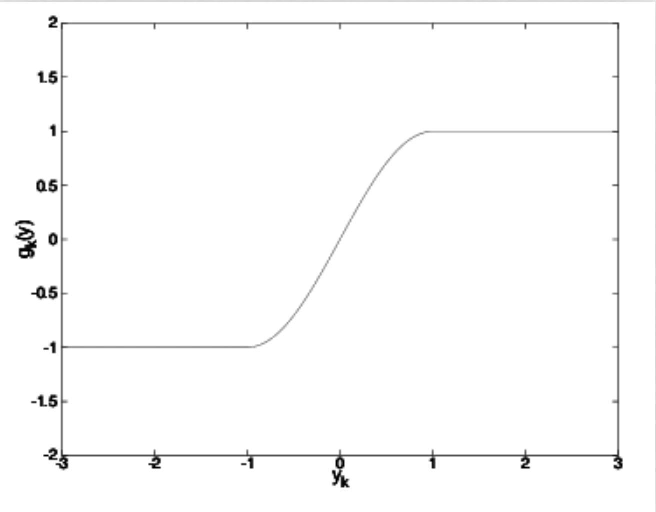


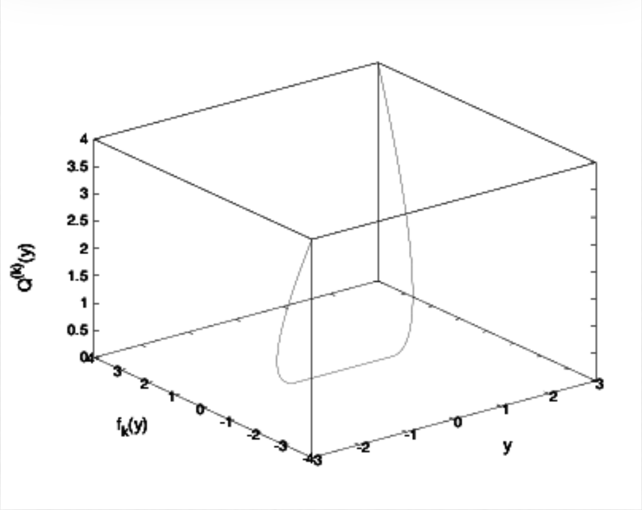


$$\nabla Q_k(\underline{\mathrm{y}}_k^{(CR)}) = J_{f_k}^T(\underline{\mathrm{y}}_k^{(CR)})g_k(\underline{\mathrm{y}}_k^{(CR)}),$$

Stationarity conditions



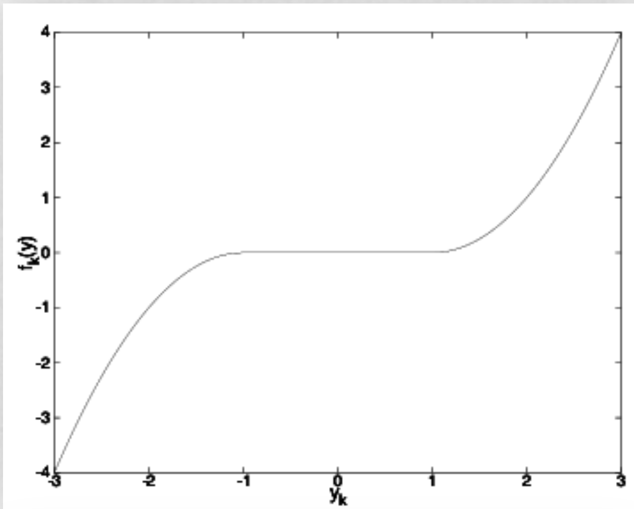


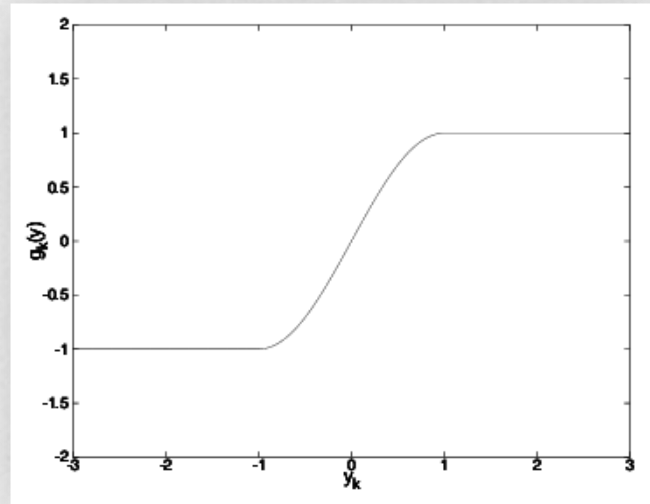


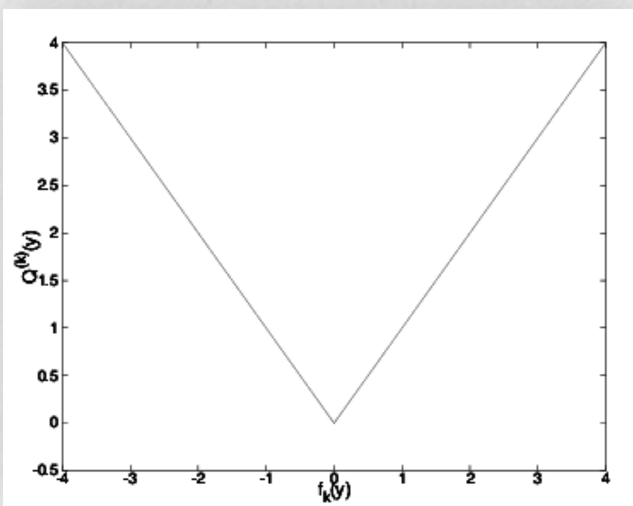
$$\nabla Q_k(\underline{\mathrm{y}}_k^{(CR)}) = J_{f_k}^T(\underline{\mathrm{y}}_k^{(CR)}) g_k(\underline{\mathrm{y}}_k^{(CR)}),$$

$$\begin{aligned} \min_{\substack{\{y_1,\ldots,y_N\} \\ \{a_1,\ldots,a_N\}}} \quad & \sum_{k=1}^{K} Q_k(\underline{\mathrm{y}}_k^{(CR)}) \\ \text{s.t.} \quad & \underline{\mathrm{a}}_k^{(CR)} = f_k(\underline{\mathrm{y}}_k^{(CR)}), \quad k = 1, \ldots, K \\ & A_\ell \underline{\mathrm{a}}_\ell^{(i)} = \underline{\mathrm{a}}_\ell^{(o)}, \quad \ell = 1, \ldots, L. \end{aligned}$$

Stationarity conditions







$$\hat{Q}_k\left(a_k^{(CR)}\right) = \left|a_k^{(CR)}\right|$$

$$a_k^{(CR)} \in \mathbf{R}$$

Stationarity conditions

Primal reduced form:

$$\begin{aligned} \min_{\{a_1,\dots,a_N\}} \quad & \sum_{k=1}^{K} \widehat{Q}_k(\underline{\mathrm{a}}_k^{(CR)}) \\ \text{s.t.} \quad & \underline{\mathrm{a}}_k^{(CR)} \in \mathcal{A}_k, \quad k = 1, \dots, K \\ & A_\ell \underline{\mathrm{a}}_\ell^{(i)} = \underline{\mathrm{a}}_\ell^{(o)}, \quad \ell = 1, \dots, L. \end{aligned}$$

$$\left\{ \begin{bmatrix} f_k(\underline{\mathrm{y}}_k^{(CR)}) \\ Q_k(\underline{\mathrm{y}}_k^{(CR)}) \end{bmatrix} : \underline{\mathrm{y}}_k^{(CR)} \in \mathbb{R}^{N_k^{(CR)}} \right\} = \left\{ \begin{bmatrix} \underline{\mathrm{a}}_k^{(CR)} \\ \widehat{Q}_k(\underline{\mathrm{a}}_k^{(CR)}) \end{bmatrix} : \underline{\mathrm{a}}_k^{(CR)} \in \mathcal{A}_k \right\}$$

Stationarity conditions

Dual reduced form:

$$
\begin{aligned}
\max_{\{b_1,\ldots,b_N\}} \quad & -\sum_{k=1}^{K} \widehat{R}_k(\underline{b}_k) \\
\text{s.t.} \quad & \underline{b}_k \in \mathcal{B}_k, \quad k = 1, \ldots, K \\
& \underline{b}_\ell^{(i)} = -A_\ell^T \underline{b}_\ell^{(o)}, \quad \ell = 1, \ldots, L,
\end{aligned}
$$

$$
\left\{ \begin{bmatrix} g_k(\underline{y}_k^{(CR)}) \\ R_k(\underline{y}_k^{(CR)}) \end{bmatrix} : \underline{y}_k^{(CR)} \in \mathbb{R}^{N_k^{(CR)}} \right\} = \left\{ \begin{bmatrix} \underline{b}_k \\ \widehat{R}_k(\underline{a}_k^{(CR)}) \end{bmatrix} : \underline{b}_k \in \mathcal{B}_k \right\}.
$$

Stationarity conditions

Primal:

$$\min_{\substack{\{y_1,\dots,y_N\}\\ \{a_1,\dots,a_N\}}} \sum_{k=1}^{K} Q_k(\underline{y}_k^{(CR)})$$
$$\text{s.t.} \quad \underline{a}_k^{(CR)} = f_k(\underline{y}_k^{(CR)}), \quad k = 1,\dots,K$$
$$A_\ell \underline{a}_\ell^{(i)} = \underline{a}_\ell^{(o)}, \quad \ell = 1,\dots,L.$$

Dual:

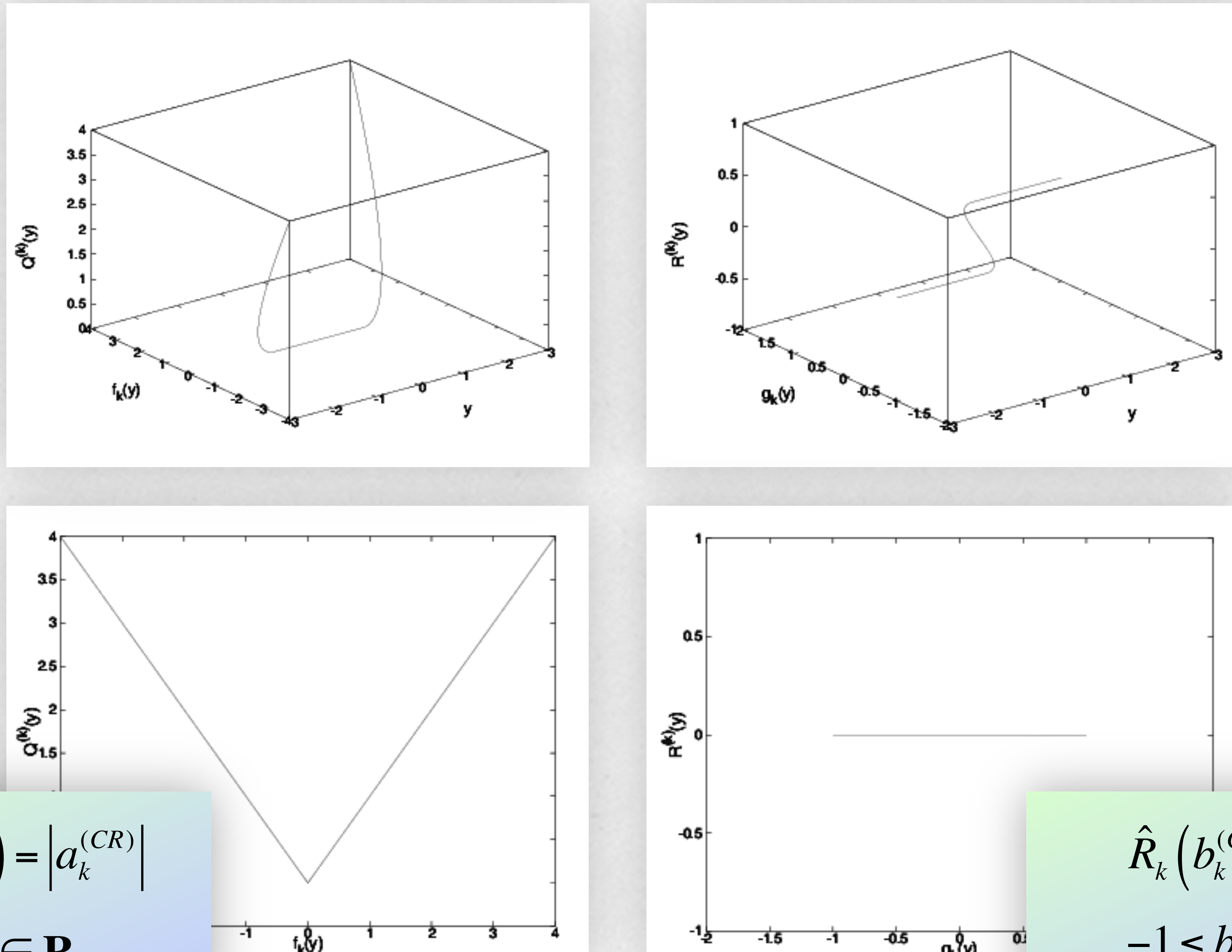
$$\max_{\substack{\{y_1,\dots,y_N\}\\ \{b_1,\dots,b_N\}}} -\sum_{k=1}^{K} R_k(\underline{y}_k^{(CR)})$$
$$\text{s.t.} \quad \underline{b}_k = g_k(\underline{y}_k^{(CR)}), \quad k = 1,\dots,K$$
$$\underline{b}_\ell^{(i)} = -A_\ell^T \underline{b}_\ell^{(o)}, \quad \ell = 1,\dots,L,$$

Conditions:

$$\underline{a}_k^{(CR)} = f_k(\underline{y}_k^{(CR)}), \quad k = 1,\dots,K$$
$$A_\ell \underline{a}_\ell^{(i)} = \underline{a}_\ell^{(o)}, \quad \ell = 1,\dots,L.$$

$$\underline{b}_k^{(CR)} = g_k(\underline{y}_k^{(CR)}), \quad k = 1,\dots,K$$
$$\underline{b}_\ell^{(i)} = -A_\ell^T \underline{b}_\ell^{(o)}, \quad \ell = 1,\dots,L,$$

Stationarity conditions



$$\hat{Q}_k\left(a_k^{(CR)}\right) = \left|a_k^{(CR)}\right|$$

$$a_k^{(CR)} \in \mathbf{R}$$

$$\hat{R}_k\left(b_k^{(CR)}\right) = 0$$

$$-1 \le b_k^{(CR)} \le 1$$

Overview

1.

Signal Processing Structures

2.

Stationarity Conditions

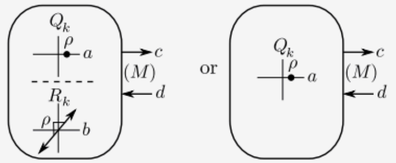
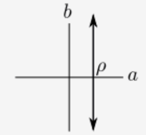
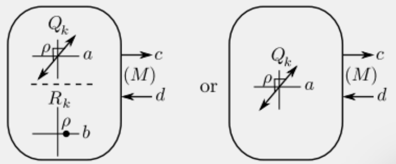
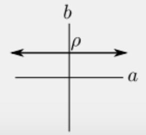
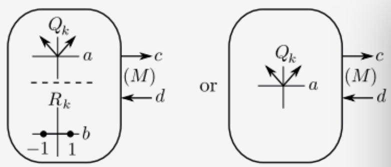
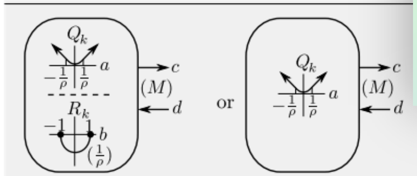
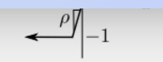
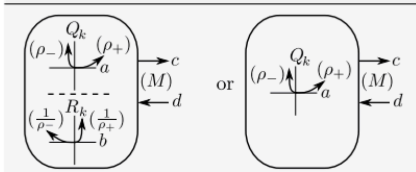
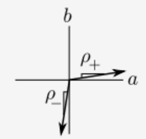
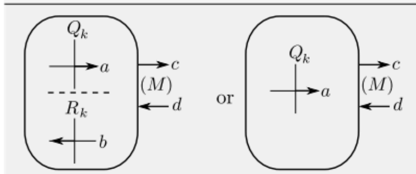
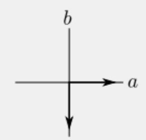
3.

Conservation

4.

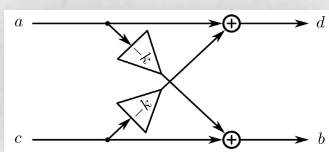
Examples

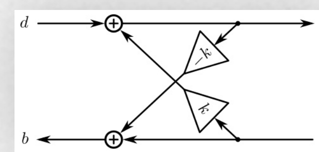
Where we’re going to end up…

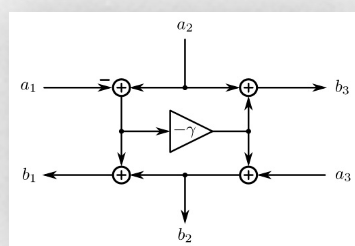
Symbol	Reduced-form primal components	Reduced-form dual components	Behavior (canonical coordinates)	Transformation matrix	Realization as a map
	$\widehat{Q}_k(a)=0$ $a=\rho$	$\widehat{R}_k(b)=\rho b$ $b\in\mathbb{R}$		$M_k^{(CR)}=\begin{bmatrix}1&-1\\1&1\end{bmatrix}$	$c=-d+2\rho$
				$M_k^{(CR)}=\begin{bmatrix}-1&1\\1&1\end{bmatrix}$	$c=d-2\rho$
	$\widehat{Q}_k(a)=\rho a$ $a\in\mathbb{R}$	$\widehat{R}_k(b)=0$ $b=\rho$		$M_k^{(CR)}=\begin{bmatrix}1&-1\\1&1\end{bmatrix}$	$c=d-2\rho$
				$M_k^{(CR)}=\begin{bmatrix}-1&1\\1&1\end{bmatrix}$	$c=-d+2\rho$
	[illegible]	[illegible]	[illegible]	[illegible]	$c=\begin{cases}d+2, & d<-1\\-d, & \lvert d\rvert\le 1\\d-2, & d>1\end{cases}$
				[illegible]	$c=\begin{cases}-d-2, & d<-1\\d, & \lvert d\rvert\le 1\\-d+2, & d>1\end{cases}$
	[illegible] $a\in\mathbb{R}$	[illegible] $-1\le b\le 1$		[illegible]	$c=\begin{cases}\frac{1-\rho}{1+\rho}d, & \lvert d\rvert\le\frac{1}{\rho}+1\\d-2, & d>\frac{1}{\rho}+1\\d+2, & d<-\frac{1}{\rho}-1\end{cases}$
				$M_k^{(CR)}=\begin{bmatrix}-1&1\\1&1\end{bmatrix}$	$c=\begin{cases}\frac{\rho-1}{\rho+1}d, & \lvert d\rvert\le\frac{1}{\rho}+1\\-d+2, & d>\frac{1}{\rho}+1\\-d-2, & d<-\frac{1}{\rho}-1\end{cases}$
	$\widehat{Q}_k(a)=\begin{cases}\frac{1}{2}\rho_+a^2 & a\ge 0\\\frac{1}{2}\rho_-a^2 & a<0\end{cases}$ $a\in\mathbb{R}$	$\widehat{R}_k(b)=\begin{cases}\frac{1}{2}\frac{1}{\rho_+}b^2 & b\ge 0\\\frac{1}{2}\frac{1}{\rho_-}b^2 & b<0\end{cases}$ $b\in\mathbb{R}$		$M_k^{(CR)}=\begin{bmatrix}1&-1\\1&1\end{bmatrix}$	$c=\begin{cases}\frac{1-\rho_+}{1+\rho_+}d, & d\ge 0\\\frac{1-\rho_-}{1+\rho_-}d, & d<0\end{cases}$
				$M_k^{(CR)}=\begin{bmatrix}-1&1\\1&1\end{bmatrix}$	$c=\begin{cases}\frac{\rho_+-1}{\rho_++1}d, & d\ge 0\\\frac{\rho_--1}{\rho_-+1}d, & d<0\end{cases}$
	$\widehat{Q}_k(a)=0$ $a\ge 0$	$\widehat{R}_k(b)=0$ $b\le 0$		$M_k^{(CR)}=\begin{bmatrix}1&-1\\1&1\end{bmatrix}$	$c=\lvert d\rvert$
				$M_k^{(CR)}=\begin{bmatrix}-1&1\\1&1\end{bmatrix}$	$c=-\lvert d\rvert$

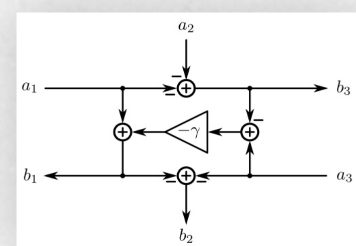
Conservation: the bridge

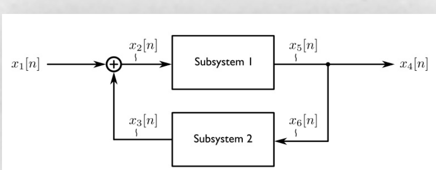
Conservation: the bridge











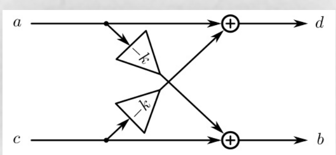
Conservation: the bridge

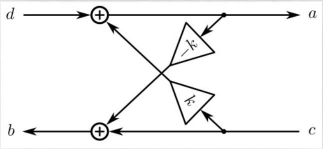
$$-2x_1[n_0]x_4[n_0] + 2x_2[n_0]x_5[n_0] + -2x_3[n_0]x_6[n_0] = 0$$

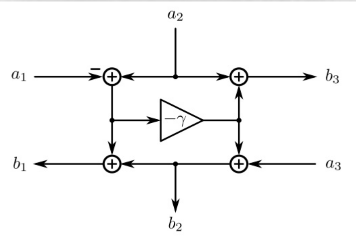
$$2\gamma(a_1^2 - b_1^2) + 2(1-\gamma)(a_2^2 - b_2^2) + 2(a_3^2 - b_3^2) = 0$$

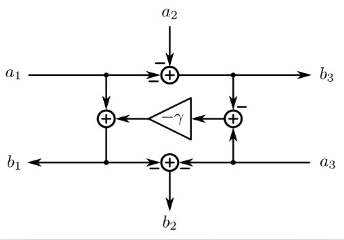
$$2(1-\gamma)(a_1^2 - b_1^2) + 2\gamma(a_2^2 - b_2^2) + 2\gamma(1-\gamma)(a_3^2 - b_3^2) = 0$$

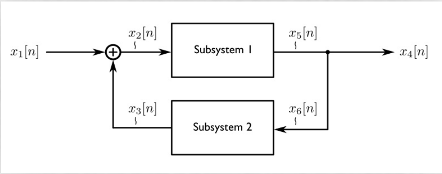
$$2(k^2 - 1)(a^2 - c^2) + 2(d^2 - b^2) = 0$$











Conservation: the bridge

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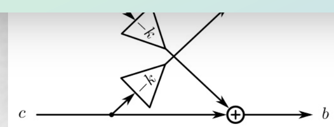
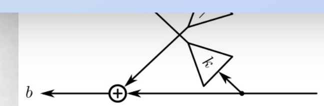
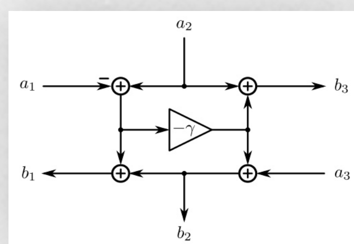
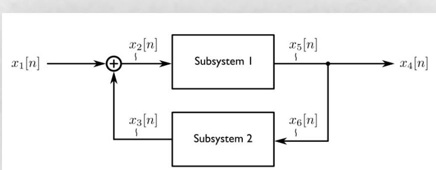
$$2(1-\gamma)(a_1^2 - b_1^2) + 2\gamma(a_2^2 - b_2^2) + 2\gamma(1-\gamma)(a_3^2 - b_3^2) = 0$$

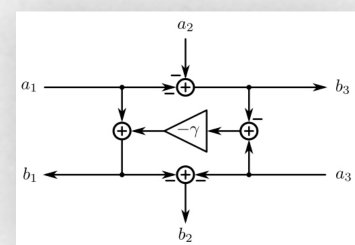
A key challenge: organizing system variables











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Conservation in Signal Processing Systems

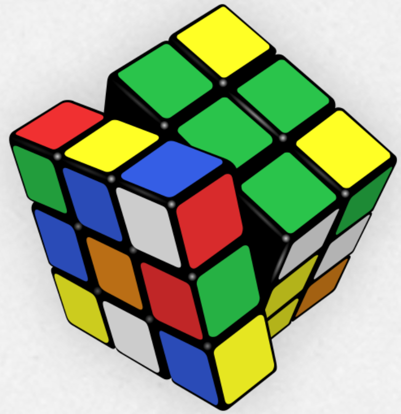
by

Thomas A. Baran

Contribution: identified isomorphism between a class of conservation principles in signal processing algorithms

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Conservation: the bridge



(group theory)

Conservation in Signal Processing Systems

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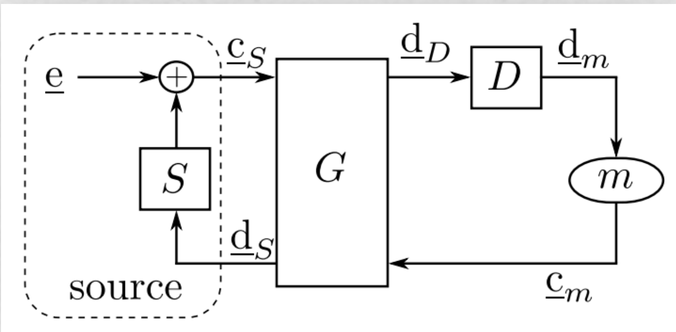
Conservation: the bridge

Conditions for stability:



$\underline{d}_m$ -> $\underline{d}_D$ is contractive

$S^T = S^{-1}$, $G^T = G^{-1}$

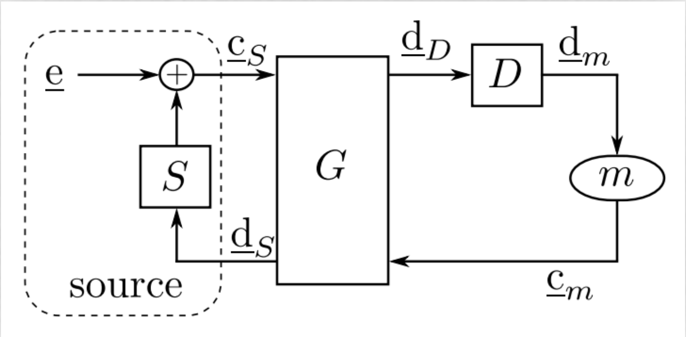
Stationarity conditions:

$$\underline{a}_k^{(CR)} = f_k(\underline{y}_k^{(CR)}), \quad k = 1, \ldots, K$$
$$A_\ell \underline{a}_\ell^{(i)} = \underline{a}_\ell^{(o)}, \quad \ell = 1, \ldots, L.$$

$$\underline{b}_k^{(CR)} = g_k(\underline{y}_k^{(CR)}), \quad k = 1, \ldots, K$$
$$\underline{b}_\ell^{(i)} = -A_\ell^T \underline{b}_\ell^{(o)}, \quad \ell = 1, \ldots, L,$$

Conservation: the bridge

Conditions for stability:



$\underline{d}_m$ -> $\underline{d}_D$ is contractive

$S^T = S^{-1}$, $G^T = G^{-1}$

$$\langle \underline{a}, \underline{b} \rangle = 0 \qquad ||\underline{d}||_2^2 - ||\underline{c}||_2^2 = 0$$

Stationarity conditions:

$$\underline{a}_k^{(CR)} = f_k(\underline{y}_k^{(CR)}), \quad k = 1, \ldots, K$$
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$$\underline{b}_\ell^{(i)} = -A_\ell^T \underline{b}_\ell^{(o)}, \quad \ell = 1, \ldots, L,$$

Conservation: the bridge

Isomorphic conservation principles:

$$\langle \underline{a}, \underline{b} \rangle = 0 \qquad\qquad ||\underline{d}||_2^2 - ||\underline{c}||_2^2 = 0$$

(Sylvester's Law of Inertia)

Conservation: the bridge

General strategy in linking structures to conditions

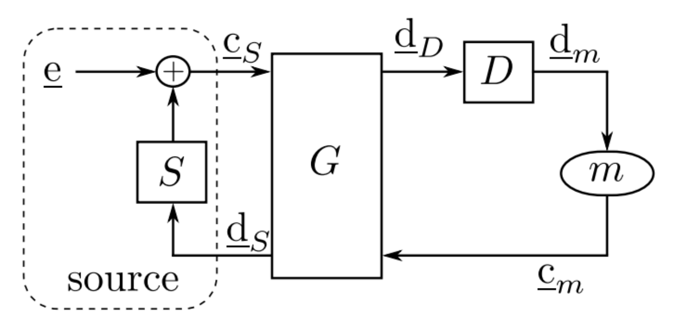
Determine a transformation M, which when applied to stationarity conditions, results in:

1. Orthogonal G
2. Contractive from $\underline{d}_m$ to $\underline{d}_D$

$$\begin{aligned} \min_{\substack{\{y_1,\ldots,y_N\} \\ \{a_1,\ldots,a_N\}}} \quad & \sum_{k=1}^{K} Q_k(\underline{y}_k^{(CR)}) \\ \text{s.t.} \quad & \underline{a}_k^{(CR)} = f_k(\underline{y}_k^{(CR)}), \quad k = 1, \ldots, K \\ & A_\ell \underline{a}_\ell^{(i)} = \underline{a}_\ell^{(o)}, \quad \ell = 1, \ldots, L. \end{aligned}$$

$$\begin{aligned} \max_{\substack{\{y_1,\ldots,y_N\} \\ \{b_1,\ldots,b_N\}}} \quad & -\sum_{k=1}^{K} R_k(\underline{y}_k^{(CR)}) \\ \text{s.t.} \quad & \underline{b}_k = g_k(\underline{y}_k^{(CR)}), \quad k = 1, \ldots, K \\ & \underline{b}_\ell^{(i)} = -A_\ell^T \underline{b}_\ell^{(o)}, \quad \ell = 1, \ldots, L, \end{aligned}$$

a and *b* variables



c and *d* variables

General strategy in linking structures to conditions

Determine a transformation *M*, which when applied to stationarity conditions, results in:

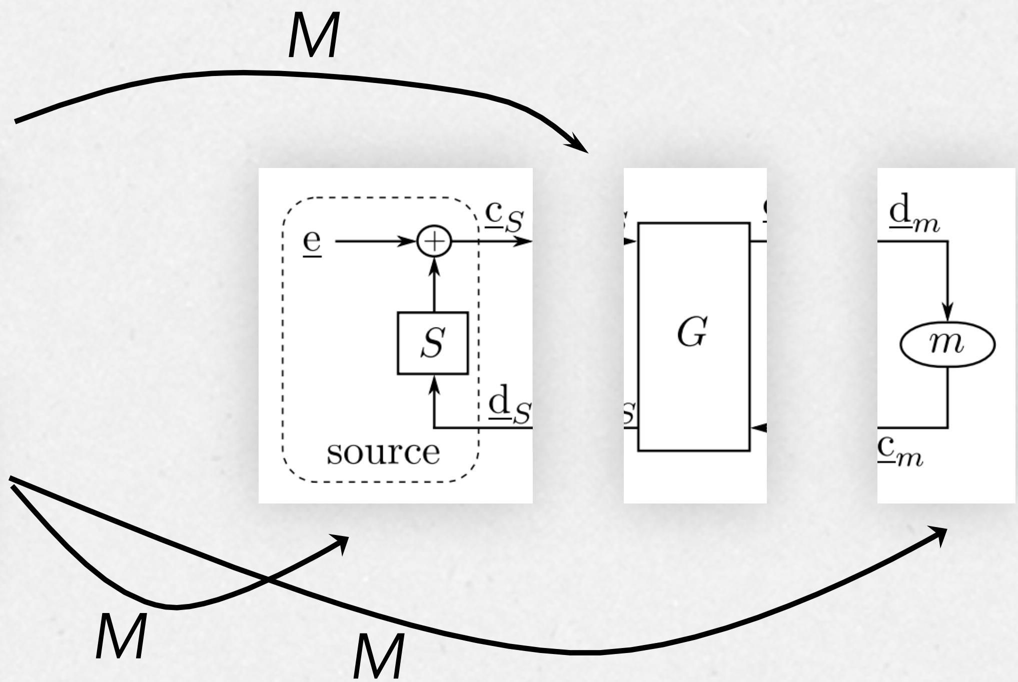
1. Orthogonal *G*
2. Contractive from $\underline{d}_m$ to $\underline{d}_D$

$$A_\ell \underline{a}_\ell^{(i)} = \underline{a}_\ell^{(o)},$$

$$\underline{b}_\ell^{(i)} = -A_\ell^T \underline{b}_\ell^{(o)},$$

$$\underline{a}_k^{(CR)} = f_k(\underline{y}_k^{(CR)}),$$

$$\underline{b}_k^{(CR)} = g_k(\underline{y}_k^{(CR)}),$$



Conservation: the bridge

General strategy in linking structures to conditions

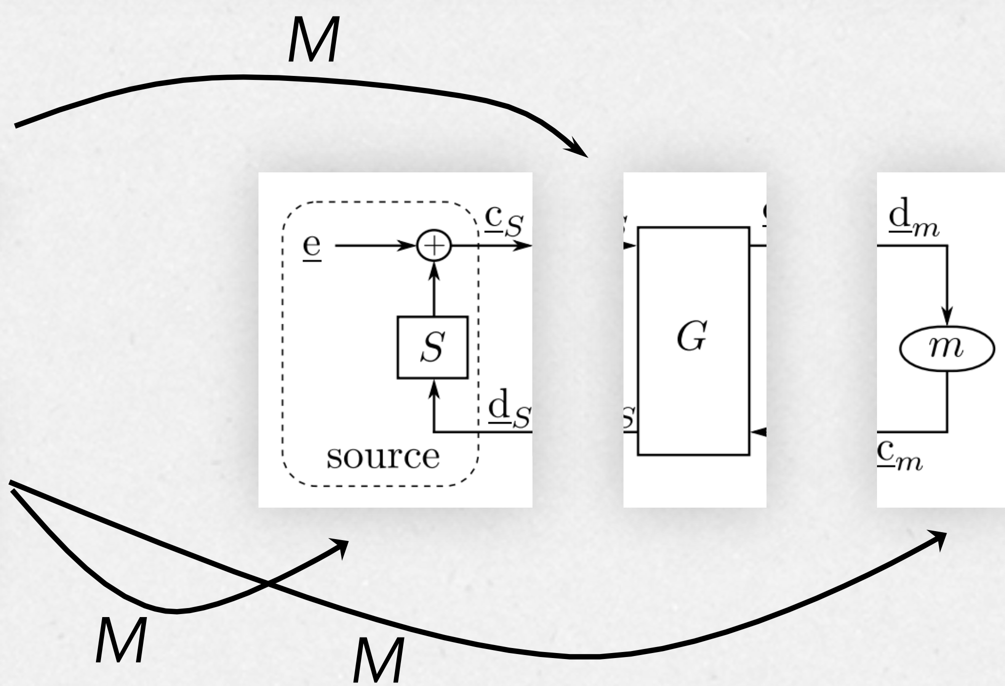
Resulting structure processes a superposition of primal and dual variables

$$A_\ell \underline{a}_\ell^{(i)} = \underline{a}_\ell^{(o)},$$

$$\underline{b}_\ell^{(i)} = -A_\ell^T \underline{b}_\ell^{(o)},$$

$$\underline{a}_k^{(CR)} = f_k(\underline{y}_k^{(CR)}),$$

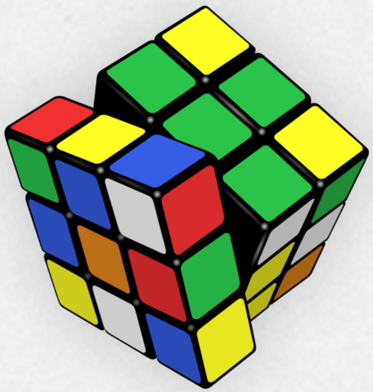
$$\underline{b}_k^{(CR)} = g_k(\underline{y}_k^{(CR)}),$$



General strategy in linking structures to conditions

Determine a transformation M, which when applied to stationarity conditions, results in:

1. Orthogonal G
2. Contractive from $\underline{d}_m$ to $\underline{d}_D$



M can, in general, depend on A_l

Conservation: the bridge

General strategy in linking structures to conditions

Determine a transformation *M*, which when applied to stationarity conditions, results in:

1. Orthogonal *G*
2. Contractive from $\underline{d}_m$ to $\underline{d}_D$

$$\underline{c}_\ell^{(i)} = \underline{a}_\ell^{(i)} - \underline{b}_\ell^{(i)} \qquad \underline{c}_\ell^{(o)} = -\underline{a}_\ell^{(o)} + \underline{b}_\ell^{(o)}$$

$$\underline{d}_\ell^{(i)} = \underline{a}_\ell^{(i)} + \underline{b}_\ell^{(i)} \qquad \underline{d}_\ell^{(o)} = \underline{a}_\ell^{(o)} + \underline{b}_\ell^{(o)}$$

A selection for *M* independent of A_l

Conservation: the bridge

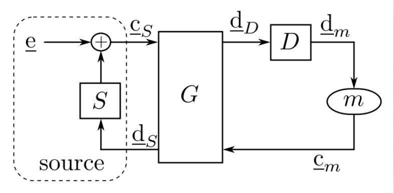
1. Orthogonal G

$$A_\ell \underline{a}_\ell^{(i)} = \underline{a}_\ell^{(o)},$$
$$\underline{b}_\ell^{(i)} = -A_\ell^T \underline{b}_\ell^{(o)},$$

→

$$G_\ell = \left(I_{N_\ell^{(LI)}} + \begin{bmatrix} 0 & -A_\ell^T \\ A_\ell & 0 \end{bmatrix}\right)\left(I_{N_\ell^{(LI)}} - \begin{bmatrix} 0 & -A_\ell^T \\ A_\ell & 0 \end{bmatrix}\right)^{-1}$$
$$G_\ell \underline{c}_\ell^{(LI)} = \underline{d}_\ell^{(LI)}$$

→

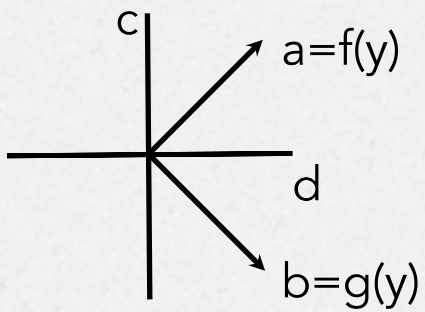


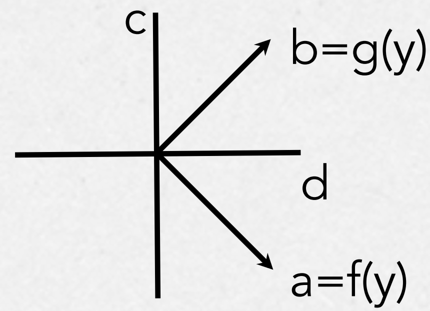
$$\underline{c}_\ell^{(i)} = \underline{a}_\ell^{(i)} - \underline{b}_\ell^{(i)} \qquad \underline{c}_\ell^{(o)} = -\underline{a}_\ell^{(o)} + \underline{b}_\ell^{(o)}$$
$$\underline{d}_\ell^{(i)} = \underline{a}_\ell^{(i)} + \underline{b}_\ell^{(i)} \qquad \underline{d}_\ell^{(o)} = \underline{a}_\ell^{(o)} + \underline{b}_\ell^{(o)}$$

A selection for *M* independent of A_l

Conservation: the bridge

2. Contractive from $\underline{d}_m$ to $\underline{d}_D$





$$\underline{c}_\ell^{(i)} = \underline{a}_\ell^{(i)} - \underline{b}_\ell^{(i)} \qquad \underline{c}_\ell^{(o)} = -\underline{a}_\ell^{(o)} + \underline{b}_\ell^{(o)}$$

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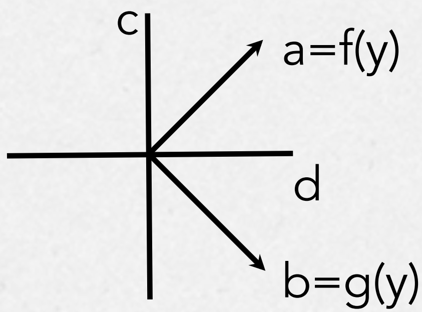
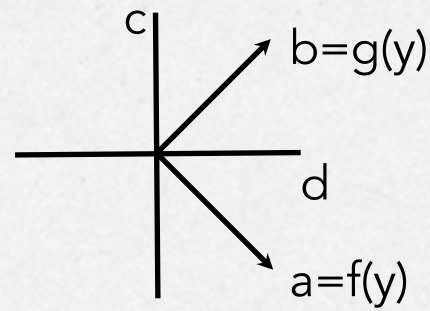
A selection for *M* independent of A_l

Conservation: the bridge

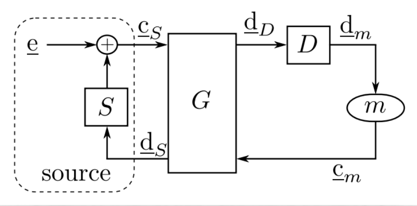
2. Contractive from $\underline{d}_m$ to $\underline{d}_D$





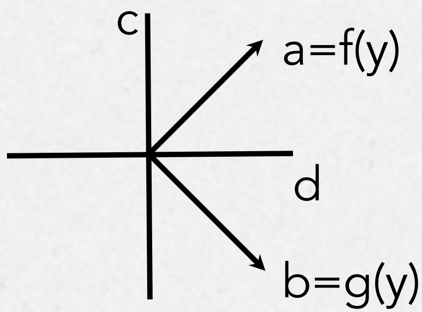
For example (contractive from $\underline{d}_m$ to $\underline{c}_m$):

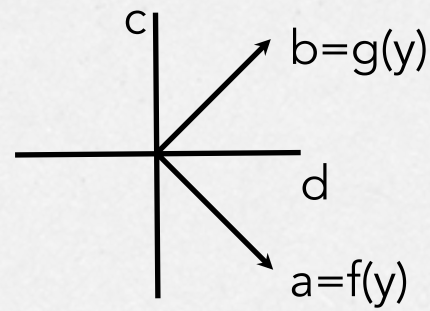
$$f_k(y_k) = y_k \qquad g_k(y_k) \qquad => \qquad Q'_k(y_k) = g_k(y_k)$$



Conservation: the bridge

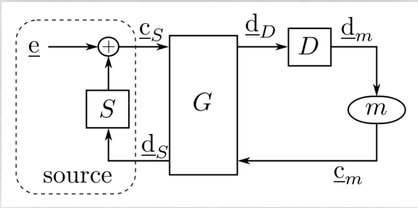
2. Contractive from $\underline{d}_m$ to $\underline{d}_D$





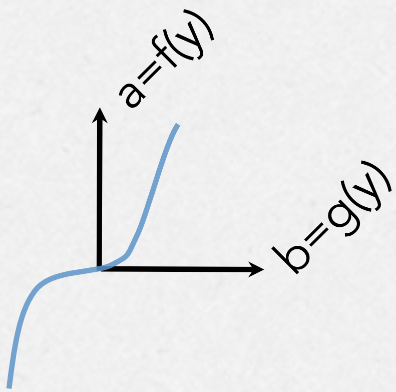
For example (contractive from $\underline{d}_m$ to $\underline{c}_m$):

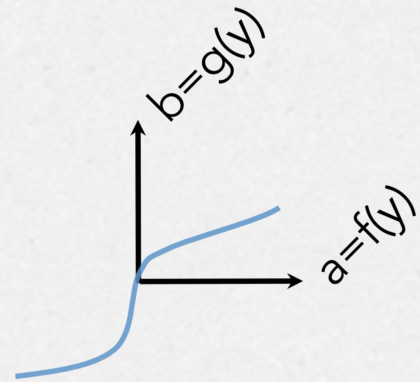
$$f_k(y_k) = y_k \qquad g_k(y_k) \qquad => \qquad Q_k'(a_k) = g_k(a_k)$$



Conservation: the bridge

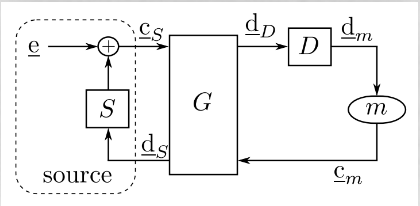
2. Contractive from $\underline{d}_m$ to $\underline{d}_D$





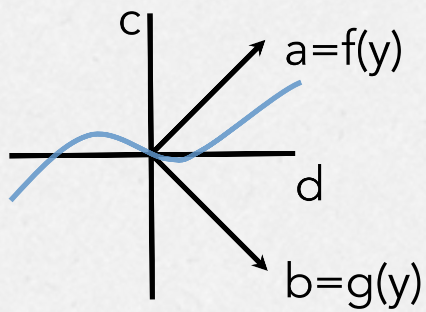
For example (contractive from $\underline{d}_m$ to $\underline{c}_m$):

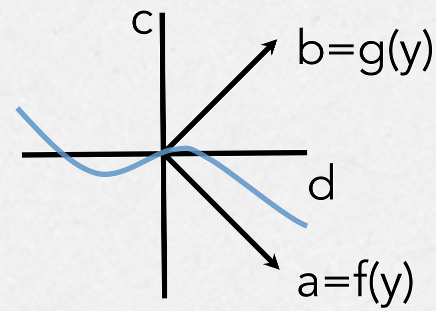
$$\eta \le Q_k''(a_k) \le \frac{1}{\eta} \qquad 0 < \eta < 1 \qquad \Rightarrow \qquad \eta \le g_k'(a_k) \le \frac{1}{\eta}$$



Conservation: the bridge

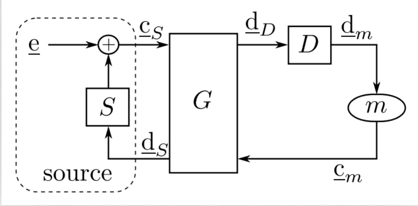
2. Contractive from $\underline{d}_m$ to $\underline{d}_D$





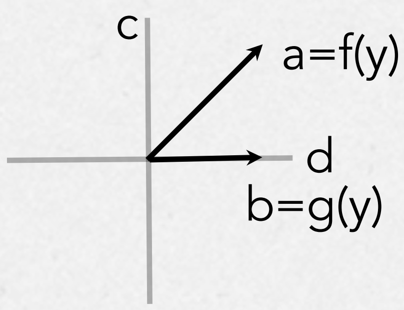
For example (contractive from $\underline{d}_m$ to $\underline{c}_m$):

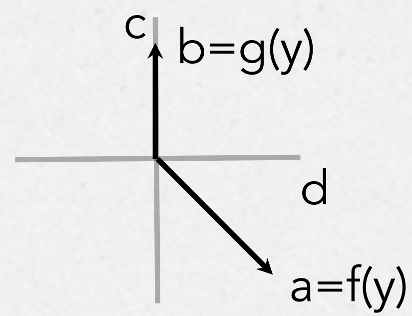
$$\eta \le Q_k''(a_k) \le \frac{1}{\eta} \qquad 0 < \eta < 1 \qquad => \qquad \left|c_k'(d_k)\right| \le \frac{1-\eta}{1+\eta}$$



Conservation: the bridge

$$A_l: \quad \underline{a}_1^{(i)} + \underline{a}_2^{(i)} = \underline{a}_3^{(o)}$$





$$\underline{c}_\ell^{(i)} = \underline{a}_\ell^{(i)}$$
$$\underline{d}_\ell^{(i)} = \underline{a}_\ell^{(i)} + \underline{b}_\ell^{(i)}$$

$$\underline{c}_\ell^{(o)} = -\underline{a}_\ell^{(o)} + \underline{b}_\ell^{(o)}$$
$$\underline{d}_\ell^{(o)} = \underline{a}_\ell^{(o)}$$

A selection for *M* utilizing structure in A_l

Overview

1.

Signal Processing Structures

2.

Stationarity Conditions

3.

Conservation

4.

Examples

Examples

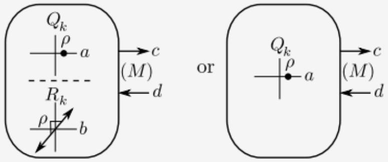
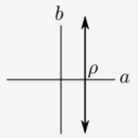
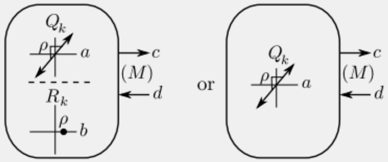
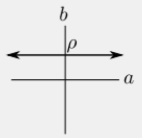
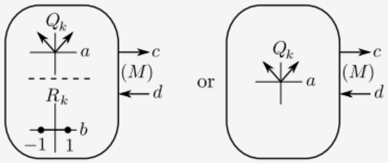
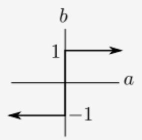
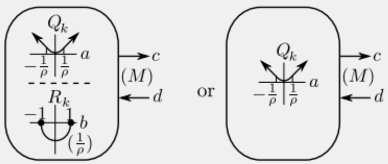
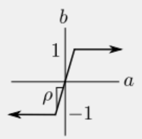
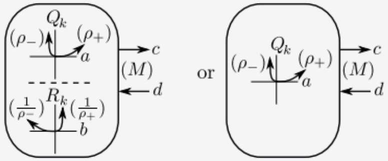
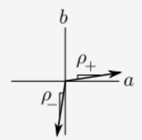
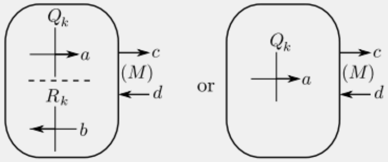
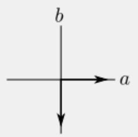
Key result: elements for asynchronous optimization

$$
\begin{aligned}
\min_{\{a_1,\dots,a_N\}} \quad & \sum_{k=1}^{K} \widehat{Q}_k(\underline{\mathbf{a}}_k^{(CR)}) \\
\text{s.t.} \quad & \underline{\mathbf{a}}_k^{(CR)} \in \mathcal{A}_k, \quad k = 1, \dots, K \\
& A_\ell \underline{\mathbf{a}}_\ell^{(i)} = \underline{\mathbf{a}}_\ell^{(o)}, \quad \ell = 1, \dots, L.
\end{aligned}
$$

$$G_\ell = \left(I_{N_\ell^{(LI)}} + \begin{bmatrix} 0 & -A_\ell^T \\ A_\ell & 0 \end{bmatrix}\right)\left(I_{N_\ell^{(LI)}} - \begin{bmatrix} 0 & -A_\ell^T \\ A_\ell & 0 \end{bmatrix}\right)^{-1}$$

$$G_\ell \underline{c}_\ell^{(LI)} = \underline{d}_\ell^{(LI)}$$

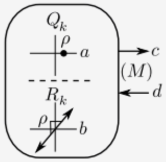
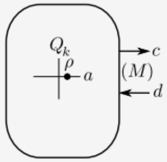
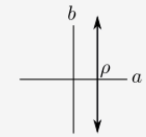
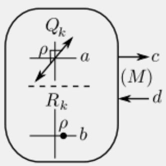
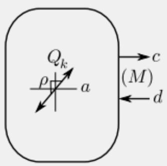
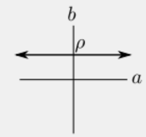
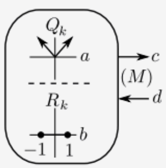
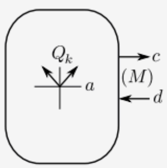
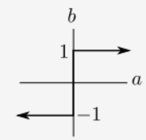
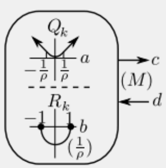
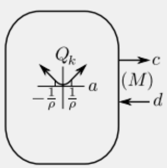
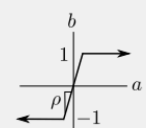
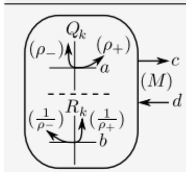
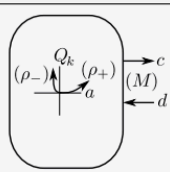
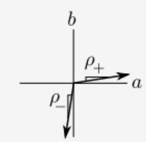
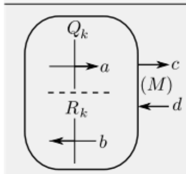
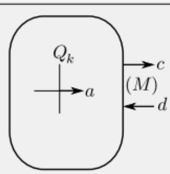
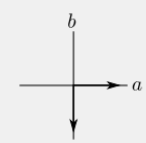
Key result: elements for asynchronous optimization

Symbol	Reduced-form primal components	Reduced-form dual components	Behavior (canonical coordinates)	Transformation matrix	Realization as a map
	$\widehat{Q}_k(a) = 0$ $a = \rho$	$\widehat{R}_k(b) = \rho b$ $b \in \mathbb{R}$		$M_k^{(CR)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$	$c = -d + 2\rho$
				$M_k^{(CR)} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	$c = d - 2\rho$
	$\widehat{Q}_k(a) = \rho a$ $a \in \mathbb{R}$	$\widehat{R}_k(b) = 0$ $b = \rho$		$M_k^{(CR)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$	$c = d - 2\rho$
				$M_k^{(CR)} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	$c = -d + 2\rho$
	$\widehat{Q}_k(a) = \lvert a \rvert$ $a \in \mathbb{R}$	$\widehat{R}_k(b) = 0$ $-1 \le b \le 1$		$M_k^{(CR)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$	$c = \begin{cases} d+2, & d < -1 \\ -d, & \lvert d \rvert \le 1 \\ d-2, & d > 1 \end{cases}$
				$M_k^{(CR)} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	$c = \begin{cases} -d-2, & d < -1 \\ d, & \lvert d \rvert \le 1 \\ -d+2, & d > 1 \end{cases}$
	$\widehat{Q}_k(a) = \begin{cases} \lvert a \rvert & \lvert a \rvert \ge \frac{1}{\rho} \\ \frac{1}{2}\rho a^2 + \frac{1}{2\rho} & \lvert a \rvert < \frac{1}{\rho} \end{cases}$ $a \in \mathbb{R}$	$\widehat{R}_k(b) = \frac{1}{2\rho} b^2 - \frac{1}{2\rho}$ $-1 \le b \le 1$		$M_k^{(CR)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$	$c = \begin{cases} \frac{1-\rho}{1+\rho} d, & \lvert d \rvert \le \frac{1}{\rho} + 1 \\ d-2, & d > \frac{1}{\rho} + 1 \\ d+2, & d < -\frac{1}{\rho} - 1 \end{cases}$
				$M_k^{(CR)} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	$c = \begin{cases} \frac{\rho-1}{\rho+1} d, & \lvert d \rvert \le \frac{1}{\rho} + 1 \\ -d+2, & d > \frac{1}{\rho} + 1 \\ -d-2, & d < -\frac{1}{\rho} - 1 \end{cases}$
	$\widehat{Q}_k(a) = \begin{cases} \frac{1}{2}\rho_+ a^2 & a \ge 0 \\ \frac{1}{2}\rho_- a^2 & a < 0 \end{cases}$ $a \in \mathbb{R}$	$\widehat{R}_k(b) = \begin{cases} \frac{1}{2}\frac{1}{\rho_+} b^2 & b \ge 0 \\ \frac{1}{2}\frac{1}{\rho_-} b^2 & b < 0 \end{cases}$ $b \in \mathbb{R}$		$M_k^{(CR)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$	$c = \begin{cases} \frac{1-\rho_+}{1+\rho_+} d, & d \ge 0 \\ \frac{1-\rho_-}{1+\rho_-} d, & d < 0 \end{cases}$
				$M_k^{(CR)} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	$c = \begin{cases} \frac{\rho_+ - 1}{\rho_+ + 1} d, & d \ge 0 \\ \frac{\rho_- - 1}{\rho_- + 1} d, & d < 0 \end{cases}$
	$\widehat{Q}_k(a) = 0$ $a \ge 0$	$\widehat{R}_k(b) = 0$ $b \le 0$		$M_k^{(CR)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$	$c = \lvert d \rvert$
				$M_k^{(CR)} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	$c = -\lvert d \rvert$

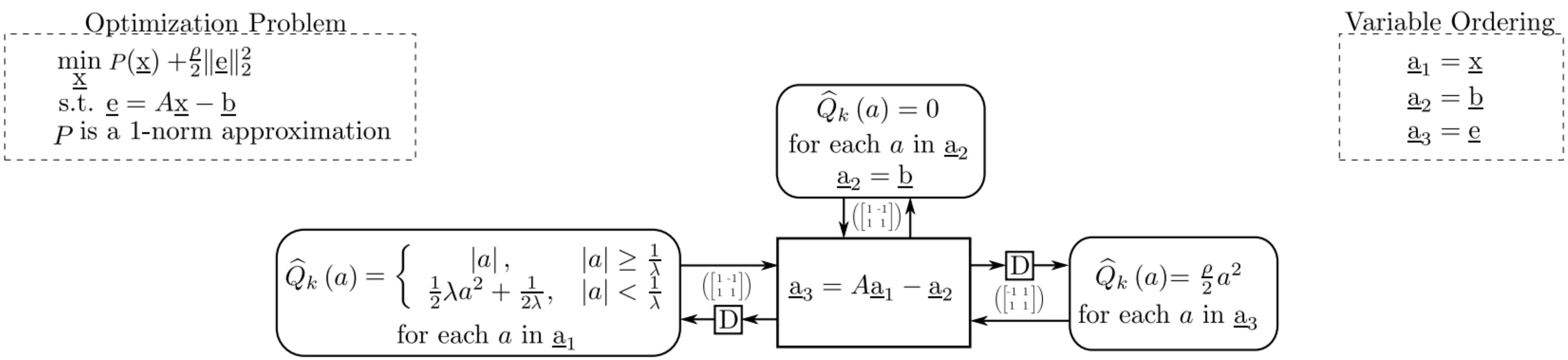
$$G_\ell = \left(I_{N_\ell^{(LI)}} + \begin{bmatrix} 0 & -A_\ell^T \\ A_\ell & 0 \end{bmatrix} \right) \left(I_{N_\ell^{(LI)}} - \begin{bmatrix} 0 & -A_\ell^T \\ A_\ell & 0 \end{bmatrix} \right)^{-1}$$

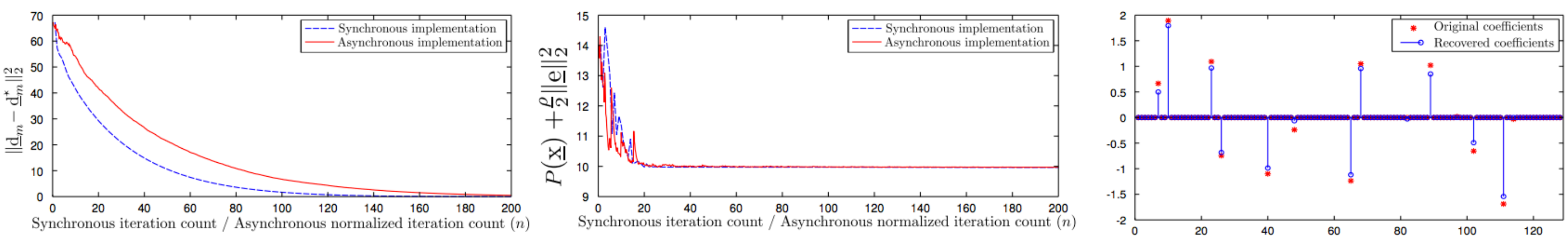
$$G_\ell \underline{c}_\ell^{(LI)} = \underline{d}_\ell^{(LI)}$$

T. A. Baran and T. A. Lahlou, "Conservative Signal Processing Architectures For Asynchronous, Distributed Optimization Part I: General Framework," in Proc. of IEEE Global Conference on Signal and Information Processing (GlobalSIP), 2014.

Symbol	Reduced-form primal components	Reduced-form dual components	Behavior (canonical coordinates)	Transformation matrix	Realization as a map
 or 	$\widehat{Q}_k(a) = 0$ $a = \rho$	$\widehat{R}_k(b) = \rho b$ $b \in \mathbb{R}$		$M_k^{(CR)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$	$c = -d + 2\rho$
				$M_k^{(CR)} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	$c = d - 2\rho$
 or 	$\widehat{Q}_k(a) = \rho a$ $a \in \mathbb{R}$	$\widehat{R}_k(b) = 0$ $b = \rho$		$M_k^{(CR)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$	$c = d - 2\rho$
				$M_k^{(CR)} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	$c = -d + 2\rho$
 or 	$\widehat{Q}_k(a) = \lvert a \rvert$ $a \in \mathbb{R}$	$\widehat{R}_k(b) = 0$ $-1 \le b \le 1$		$M_k^{(CR)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$	$c = \begin{cases} d+2, & d < -1 \\ -d, & \lvert d \rvert \le 1 \\ d-2, & d > 1 \end{cases}$
				$M_k^{(CR)} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	$c = \begin{cases} -d-2, & d < -1 \\ d, & \lvert d \rvert \le 1 \\ -d+2, & d > 1 \end{cases}$
 or 	$\widehat{Q}_k(a) = \begin{cases} \lvert a \rvert & \lvert a \rvert \ge \frac{1}{\rho} \\ \frac{1}{2}\rho a^2 + \frac{1}{2\rho} & \lvert a \rvert < \frac{1}{\rho} \end{cases}$ $a \in \mathbb{R}$	$\widehat{R}_k(b) = \frac{1}{2\rho} b^2 - \frac{1}{2\rho}$ $-1 \le b \le 1$		$M_k^{(CR)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$	$c = \begin{cases} \frac{1-\rho}{1+\rho} d, & \lvert d \rvert \le \frac{1}{\rho} + 1 \\ d-2, & d > \frac{1}{\rho} + 1 \\ d+2, & d < -\frac{1}{\rho} - 1 \end{cases}$
				$M_k^{(CR)} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	$c = \begin{cases} \frac{\rho-1}{\rho+1} d, & \lvert d \rvert \le \frac{1}{\rho} + 1 \\ -d+2, & d > \frac{1}{\rho} + 1 \\ -d-2, & d < -\frac{1}{\rho} - 1 \end{cases}$
 or 	$\widehat{Q}_k(a) = \begin{cases} \frac{1}{2}\rho_+ a^2 & a \ge 0 \\ \frac{1}{2}\rho_- a^2 & a < 0 \end{cases}$ $a \in \mathbb{R}$	$\widehat{R}_k(b) = \begin{cases} \frac{1}{2}\frac{1}{\rho_+} b^2 & b \ge 0 \\ \frac{1}{2}\frac{1}{\rho_-} b^2 & b < 0 \end{cases}$ $b \in \mathbb{R}$		$M_k^{(CR)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$	$c = \begin{cases} \frac{1-\rho_+}{1+\rho_+} d, & d \ge 0 \\ \frac{1-\rho_-}{1+\rho_-} d, & d < 0 \end{cases}$
				$M_k^{(CR)} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	$c = \begin{cases} \frac{\rho_+-1}{\rho_++1} d, & d \ge 0 \\ \frac{\rho_--1}{\rho_-+1} d, & d < 0 \end{cases}$
 or 	$\widehat{Q}_k(a) = 0$ $a \ge 0$	$\widehat{R}_k(b) = 0$ $b \le 0$		$M_k^{(CR)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$	$c = \lvert d \rvert$
				$M_k^{(CR)} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	$c = -\lvert d \rvert$

Examples





T. A. Baran and T. A. Lahlou, "Conservative Signal Processing Architectures For Asynchronous, Distributed Optimization Part II: Example Systems," in Proc. of IEEE Global Conference on Signal and Information Processing (GlobalSIP), 2014.

Examples

Optimization Problem

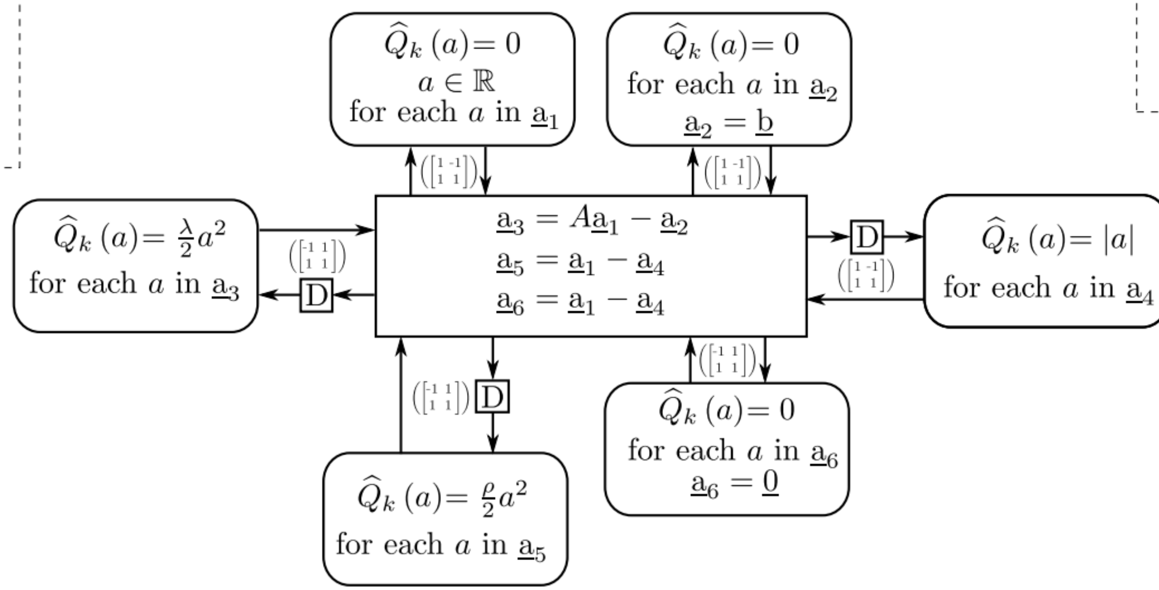
$$\min_{\underline{x}} \|\underline{z}\|_1 + \frac{\lambda}{2}\|\underline{e}\|_2^2 + \frac{\rho}{2}\|\underline{w}\|_2^2$$
$$\text{s.t. } \underline{e} = A\underline{x} - \underline{b}$$
$$\underline{w} = \underline{x} - \underline{z}$$
$$\underline{y} = \underline{x} - \underline{z}$$
$$\underline{y} = \underline{0}$$

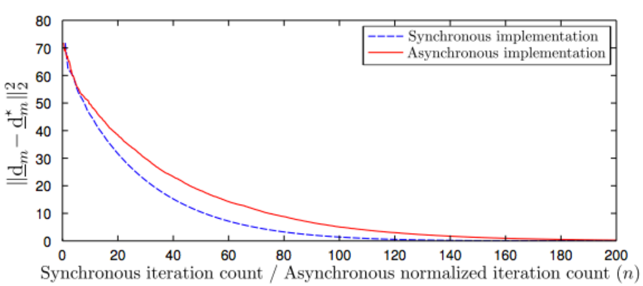
Variable Ordering

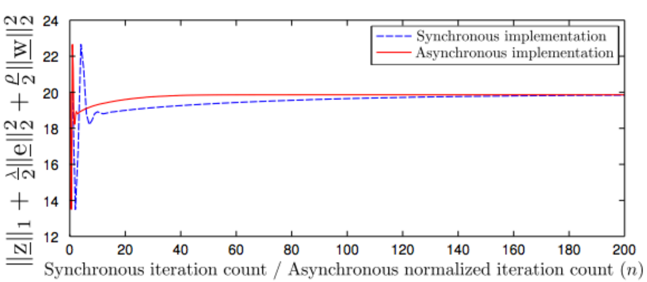
$\underline{a}_1 = \underline{x}$ $\quad \underline{a}_5 = \underline{w}$

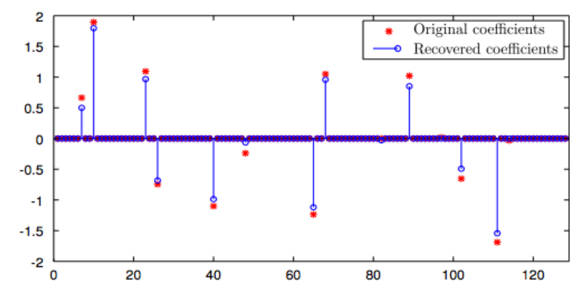
$\underline{a}_2 = \underline{b}$ $\quad \underline{a}_4 = \underline{z}$

$\underline{a}_3 = \underline{e}$ $\quad \underline{a}_6 = \underline{y}$









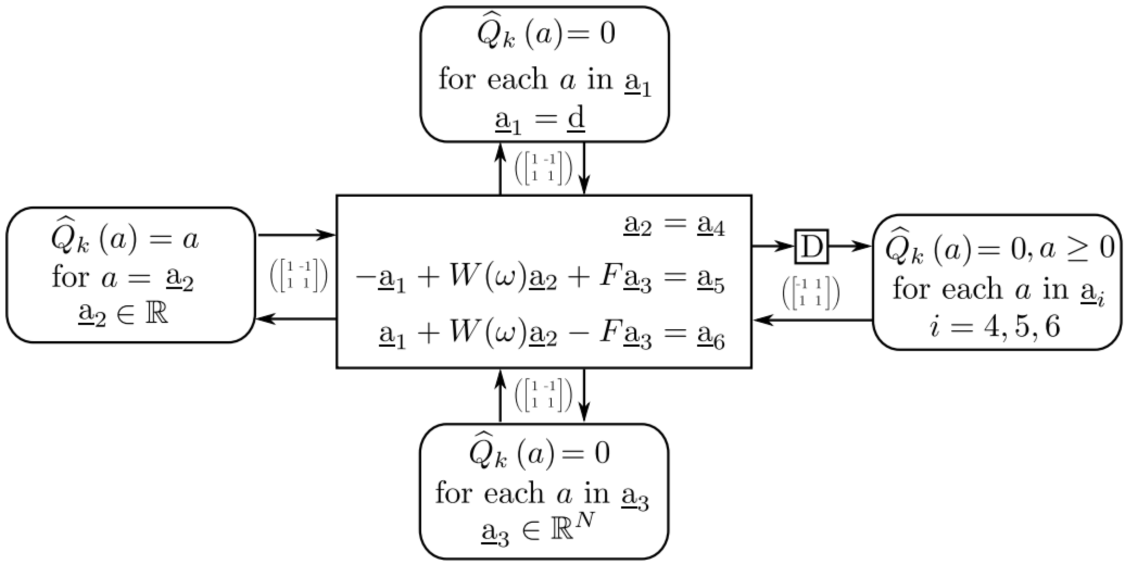
Examples

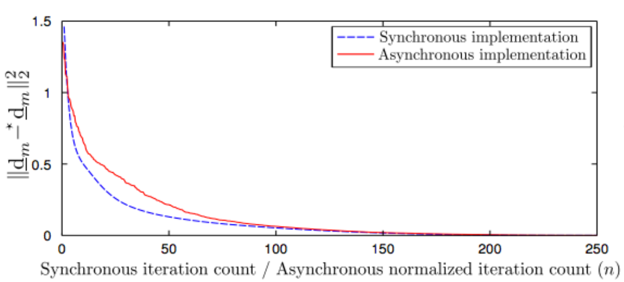
Optimization Problem

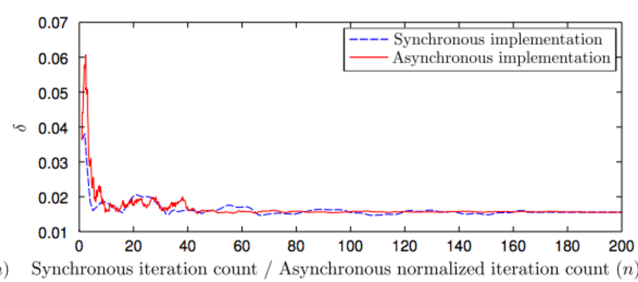
$$\min_{\delta, \underline{h}} \ \delta$$
$$\text{s.t. } |F\underline{h} - \underline{d}| \leq W(\omega)\delta$$
$$\delta \geq 0$$

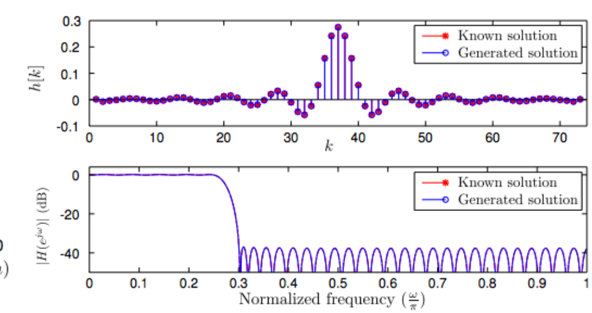
Variable Ordering

$\underline{a}_1 = \underline{d}$
$\underline{a}_2 = [\delta]$
$\underline{a}_3 = \underline{h}$

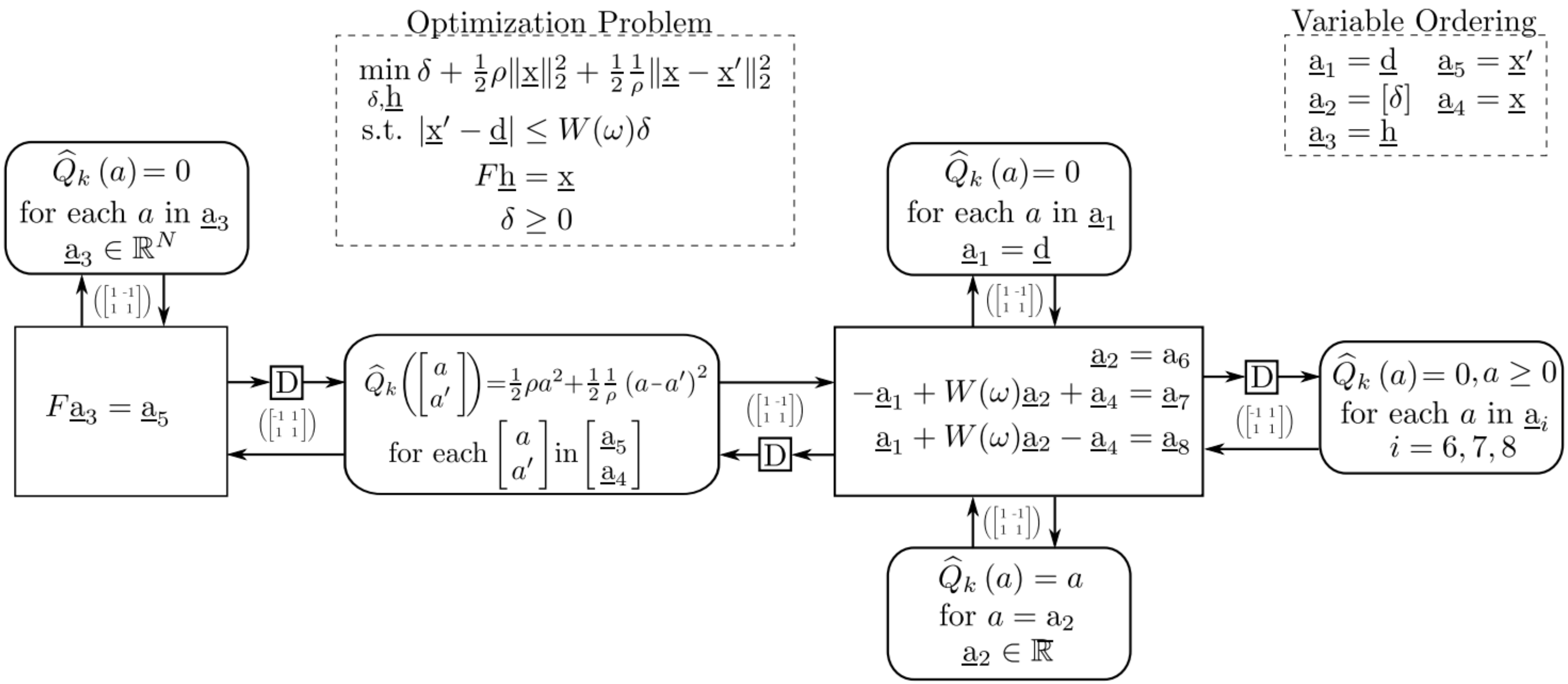




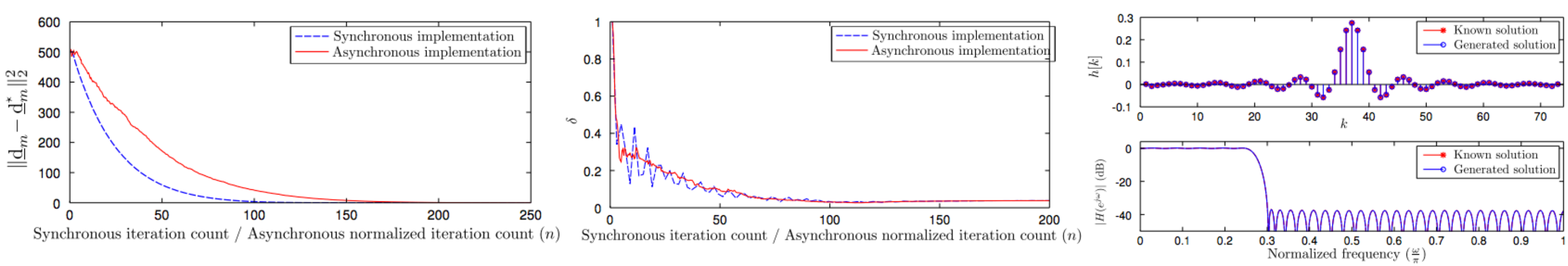




Examples

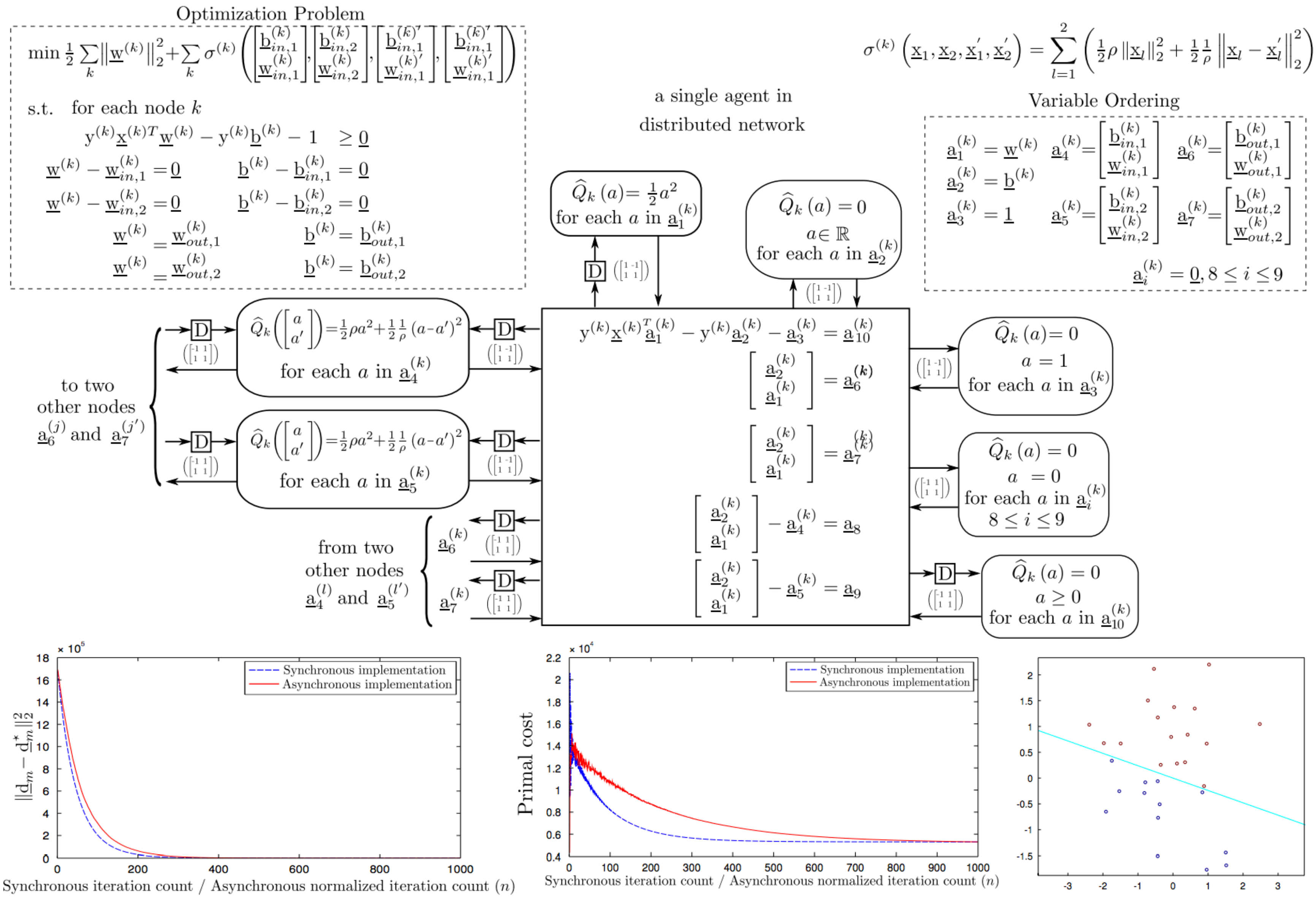


Optimization Problem
$\min_{\delta,\underline{h}} \delta + \frac{1}{2}\rho\|\underline{x}\|_2^2 + \frac{1}{2}\frac{1}{\rho}\|\underline{x} - \underline{x}'\|_2^2$
s.t. $|\underline{x}' - \underline{d}| \leq W(\omega)\delta$
$F\underline{h} = \underline{x}$
$\delta \geq 0$
Variable Ordering
$\underline{a}_1 = \underline{d}$ $\underline{a}_5 = \underline{x}'$
$\underline{a}_2 = [\delta]$ $\underline{a}_4 = \underline{x}$
$\underline{a}_3 = \underline{h}$
$\widehat{Q}_k(a) = 0$ for each a in $\underline{a}_3$, $\underline{a}_3 \in \mathbb{R}^N$
$F\underline{a}_3 = \underline{a}_5$
D
$\widehat{Q}_k\left(\begin{bmatrix} a \\ a' \end{bmatrix}\right) = \frac{1}{2}\rho a^2 + \frac{1}{2}\frac{1}{\rho}(a - a')^2$ for each $\begin{bmatrix} a \\ a' \end{bmatrix}$ in $\begin{bmatrix} \underline{a}_5 \\ \underline{a}_4 \end{bmatrix}$
$\widehat{Q}_k(a) = 0$ for each a in $\underline{a}_1$, $\underline{a}_1 = \underline{d}$
$\underline{a}_2 = \underline{a}_6$
$-\underline{a}_1 + W(\omega)\underline{a}_2 + \underline{a}_4 = \underline{a}_7$
$\underline{a}_1 + W(\omega)\underline{a}_2 - \underline{a}_4 = \underline{a}_8$
$\widehat{Q}_k(a) = 0, a \geq 0$ for each a in $\underline{a}_i$, $i = 6, 7, 8$
$\widehat{Q}_k(a) = a$ for $a = \underline{a}_2$, $\underline{a}_2 \in \mathbb{R}$



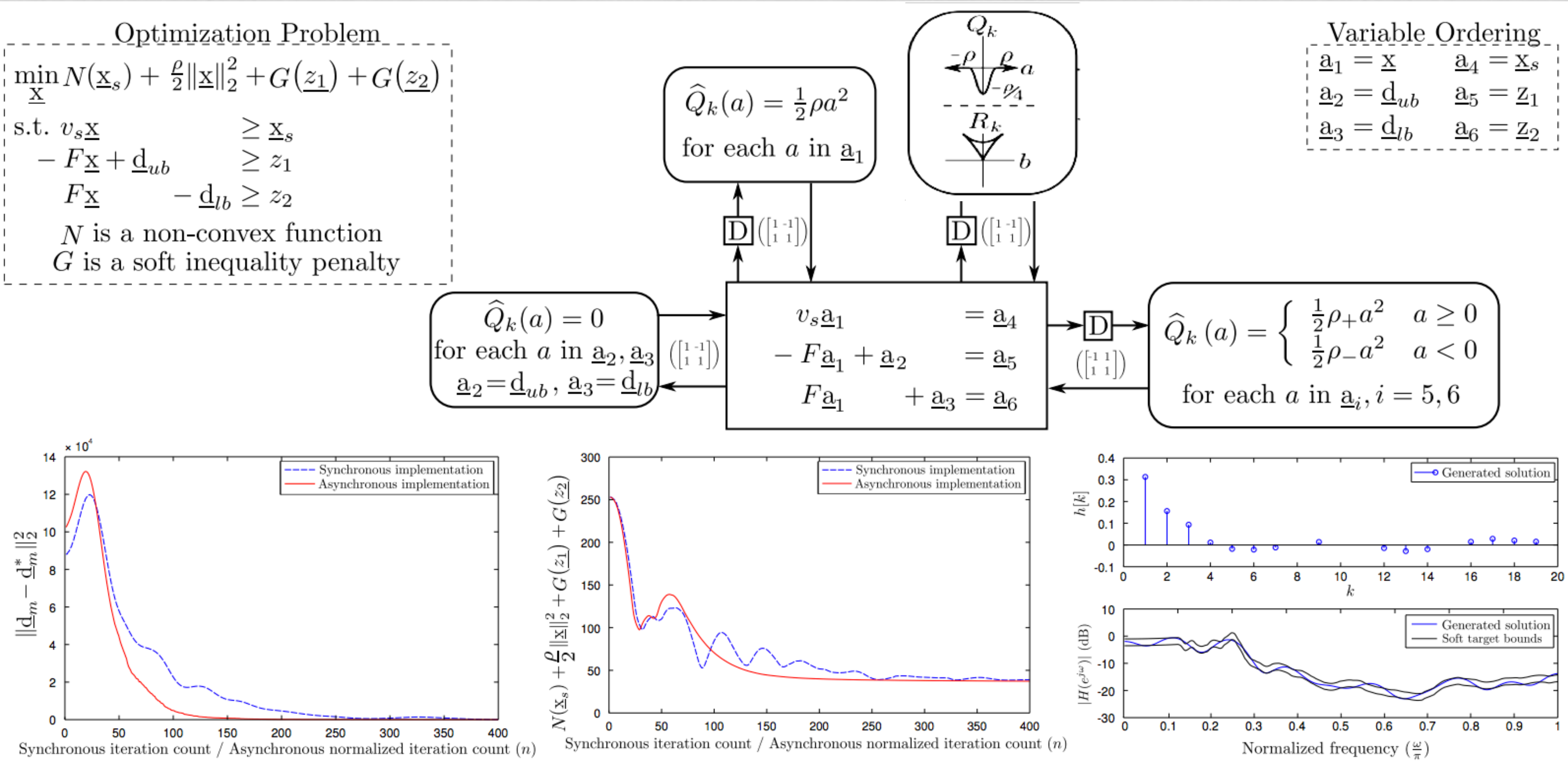
Synchronous implementation
Asynchronous implementation
$\|\underline{d}_m - \underline{d}_m^*\|_2^2$
Synchronous iteration count / Asynchronous normalized iteration count (n)
δ
Known solution
Generated solution
$h[k]$
k
Normalized frequency ($\frac{\omega}{\pi}$)

Examples



Optimization Problem
s.t. for each node k
a single agent in distributed network
Variable Ordering
to two other nodes
from two other nodes
for each a in
Synchronous implementation
Asynchronous implementation
Primal cost
Synchronous iteration count / Asynchronous normalized iteration count (n)

Examples


Optimization Problem
$\min_{\underline{x}} N(\underline{x}_s) + \frac{\rho}{2}\|\underline{x}\|_2^2 + G(\underline{z_1}) + G(\underline{z_2})$
s.t. $v_s\underline{x} \geq \underline{x}_s$
$-F\underline{x} + \underline{d}_{ub} \geq z_1$
$F\underline{x} - \underline{d}_{lb} \geq z_2$
N is a non-convex function
G is a soft inequality penalty
$\widehat{Q}_k(a) = \frac{1}{2}\rho a^2$
for each a in $\underline{a}_1$
Q_k
R_k
Variable Ordering
$\underline{a}_1 = \underline{x}$, $\underline{a}_4 = \underline{x}_s$
$\underline{a}_2 = \underline{d}_{ub}$, $\underline{a}_5 = \underline{z}_1$
$\underline{a}_3 = \underline{d}_{lb}$, $\underline{a}_6 = \underline{z}_2$
$\widehat{Q}_k(a) = 0$
for each a in $\underline{a}_2, \underline{a}_3$
$\underline{a}_2 = \underline{d}_{ub}$, $\underline{a}_3 = \underline{d}_{lb}$
$v_s\underline{a}_1 = \underline{a}_4$
$-F\underline{a}_1 + \underline{a}_2 = \underline{a}_5$
$F\underline{a}_1 + \underline{a}_3 = \underline{a}_6$
$\widehat{Q}_k(a) = \begin{cases} \frac{1}{2}\rho_+ a^2 & a \geq 0 \\ \frac{1}{2}\rho_- a^2 & a < 0 \end{cases}$
for each a in $\underline{a}_i, i = 5, 6$
D
Synchronous implementation
Asynchronous implementation
$\|\underline{d}_m - \underline{d}_m^*\|_2^2$
× 10^4
Synchronous iteration count / Asynchronous normalized iteration count (n)
$N(\underline{x}_s) + \frac{\rho}{2}\|\underline{x}\|_2^2 + G(z_1) + G(z_2)$
Generated solution
$h[k]$
k
Soft target bounds
$|H(e^{j\omega})|$ (dB)
Normalized frequency ($\frac{\omega}{\pi}$)

Examples

Conservative Signal Processing Structures For Optimization

1. Enter parameters.

Random example

A
```
[[ -1,  0,  -1,  1,  3 ],
 [  0,  1,  -2,  1,  1 ],
 [  1,  0,   4,  1,  3 ]]
```

$\underline{b}$ [4, 4, 4]

ρ_x 1000 ρ_e 2

2. Initialize system.

Initialize

```
myLasso = new BLLasso(A, b, {'rho_x': rho_x, 'rho_e': rho_e});
```

3. Process.

Start Stop Step

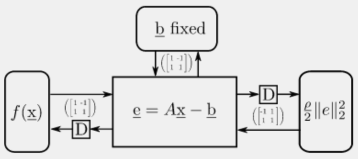
N 1 p 1

```
myLasso.process(N,p);
```

Reset

```
myLasso.reset();
```

Signal-Flow Architecture



Optimization Problem

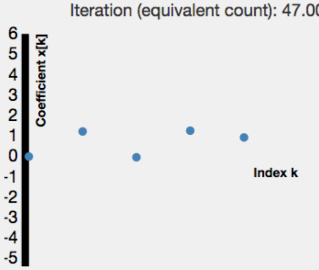
$$\min_{\underline{x}} f(\underline{x}) + \frac{\rho_e}{2} \|\underline{e}\|_2^2$$
$$\text{s.t.} \quad \underline{e} = A\underline{x} - \underline{b}$$

where f approximates the 1-norm, i.e.

$$f(a) = \begin{cases} |a|, & |a| \geq \frac{1}{\rho_x} \\ \frac{1}{2}\rho_x a^2 + \frac{1}{2\rho_x}, & |a| < \frac{1}{\rho_x} \end{cases}$$

Current Solution

Iteration (equivalent count): 47.00




```
x = myLasso.readout();
```

Examples

Conservative Signal Processing Structures For Optimization

1. Enter parameters.

Random example

A	[[-1, 0, -1, 1, 3], [0, 1, -2, 1, 1], [1, 0, 4, 1, 3]]
$\underline{b}$	[4, 4, 4]
ρ_e	2

2. Initialize system.

Initialize

```
myZeroNorm = new BLZeroNorm(A, b, rho_e);
```

3. Process.

Start Stop Step

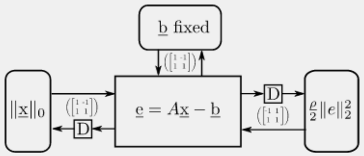
N	100	p	0.01

```
myZeroNorm.process(N,p);
```

Reset

```
myZeroNorm.reset();
```

Signal-Flow Architecture



Optimization Problem

$$\min_{\underline{x}} f(\underline{x}) + \frac{\rho_e}{2} \|\underline{e}\|_2^2$$

$$\text{s.t.} \quad \underline{e} = A\underline{x} - \underline{b}$$

where f approximates the 0-norm

Current Solution

Iteration (equivalent count): 174.00


```
x = myZeroNorm.readout();
```

Key results

A straightforward method for creating a class of signal processing structures for optimization

1. Write an optimization problem.
2. Combine specific associated elements (e.g. from table).
3. Implement synchronously or asynchronously.
4. Read out.

Key results

A strategy for determining additional signal-flow elements

1. Write component of stationarity condition.
2. Identify conservation principle.
3. Transform to obtain contractive system with $||\underline{\mathrm{d}}||_2^2 - ||\underline{\mathrm{c}}||_2^2 = 0$.

Comments

When writing asynchronous optimization algorithms

If primal and dual variables are being passed around, may want to do something differently:

Identify stationarity conditions and conservation principle.

Modify the algorithm to operate on a linear superposition of primal and dual variables.

Signal processing platforms keep evolving

Think creatively about designing algorithms to use commodity, high-performance platforms

Thank you!